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Cauchy type functional equations related to some associative rational functions

Abstract. L. Losonczi [4] determined local solutions of the generalized Cauchy equation $f(F(x, y)) = f(x) + f(y)$ on components of the definition of a given associative rational function F . The class of the associative rational function was described by A. Chéritat [1] and his work was followed by paper [3] of the author. The aim of the present paper is to describe local solutions of the equation considered for some singular associative rational functions.

1. Introduction

By an associative function on a nonempty set A usually we understand a binary operation $F: A \times A \rightarrow A$ satisfying for all $x, y, z \in A$ an equation

$$F(x, F(y, z)) = F(F(x, y), z). \quad (1)$$

Rational functions, which are defined as elements of a field of fractions of polynomials in two variables have a form of a quotient of two polynomials in two variables. Since the natural domain of such a function is usually not a rectangular $A \times A$, the associativity is defined by a conditional form of (1). We propose the following definition.

DEFINITION 1

A rational function $F: D \rightarrow \mathbb{R}$, where $D \subset \mathbb{R}^2$ is a given nonempty set is called *associative* if and only if it satisfies

$$F(x, F(y, z)) = F(F(x, y), z)$$

for all $(x, y, z) \in \mathbb{R}^3$ such that $(x, y), (y, z), (x, F(y, z)), (F(x, y), z) \in D$. An associative rational function is often called an *associative operation*.

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In [3] (Theorem 2) are included, among others, as associative the following functions (with natural domains in question)

$$F(x, y) = \frac{xy + \alpha}{x + y + \beta}, \quad (2)$$

where $\alpha, \beta \in \mathbb{R}$.

Let us consider the functional equation

$$f(F(x, y)) = f(x) + f(y), \quad (\text{E})$$

where $f: I \rightarrow \mathbb{R}$ is an unknown function, $I \subset \mathbb{R}$ is an interval, $F: D \rightarrow I$ is a given rational associative function and $(x, y) \in D \subset \mathbb{R}^2$.

In [4] the local solutions of the generalized Cauchy equation (E) were determined by L. Losonczi for the following operations of the class of operations of the form (2),

$$F(x, y) = \frac{xy + 1}{x + y},$$

$$F(x, y) = \frac{xy - 1}{x + y}.$$

They are so-called local solutions only, i.e. such that equation (E) is satisfied not necessarily on the whole domain of the definition of the operation F but only on some its subsets. For example, the components of the domain D might happen to be such subsets although, in some cases, even they may prove to be "to large". The aim of the present paper is to describe local solutions of equations

$$f\left(\frac{xy}{x + y + \beta}\right) = f(x) + f(y), \quad (\text{E1})$$

$$f\left(\frac{xy + \alpha}{x + y}\right) = f(x) + f(y), \quad (\text{E2})$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$.

2. Main results

We will start with (E1) and its trivial solutions. First we will consider the case $\beta \neq 0$.

THEOREM 1

The only solution $f: \mathbb{R} \rightarrow \mathbb{R}$ of the functional equation

$$f\left(\frac{xy}{x + y + \beta}\right) = f(x) + f(y), \quad x + y + \beta \neq 0 \quad (\text{E1.1})$$

is the constant function $f(x) = 0$ for $x \in \mathbb{R}$.

Proof. Assume that $f: \mathbb{R} \rightarrow \mathbb{R}$ is a solution of (E1.1). Substituting $y = 0$ in (E1.1) we obtain

$$f(0) = f(x) + f(0), \quad x + \beta \neq 0.$$

This means that f is of the form

$$f(x) = \begin{cases} 0 & \text{for } x \neq -\beta, \\ c & \text{for } x = -\beta, \end{cases}$$

where $c \in \mathbb{R}$. By setting $x = y = -\beta$ in (E1.1) we have

$$f\left(\frac{\beta^2}{-\beta}\right) = f(-\beta) + f(-\beta),$$

whence

$$f(-\beta) = 0$$

and, therefore, $c = 0$. This means that $f \equiv 0$. Obviously, the constant function $f \equiv 0$ is a solution of (E1.1).

A description of the solutions of (E1) in the case $\beta = 0$ is given by the following

THEOREM 2

If a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is a solution of the functional equation

$$f\left(\frac{xy}{x+y}\right) = f(x) + f(y), \quad x + y \neq 0, \quad (3)$$

then there exists a constant $c \in \mathbb{R}$ such that

$$f(x) = \begin{cases} 0 & \text{for } x \neq 0, \\ c & \text{for } x = 0. \end{cases} \quad (4)$$

Conversely, for any constant $c \in \mathbb{R}$ the above function satisfies (3).

Proof. Assume that $f: \mathbb{R} \rightarrow \mathbb{R}$ is a solution of (3). Let $x \neq 0$ be arbitrary fixed and let $y = 0$, then by (3) we obtain

$$f(0) = f(x) + f(0), \quad x \neq 0.$$

This means that f is of the form (4), where $c \in \mathbb{R}$.

In order to check that f given by (4) satisfies equation (3) fix arbitrarily a pair (x, y) such that $x \neq -y$ and consider the following three cases

- (i) $x \neq 0, y = 0$,
- (ii) $x = 0, y \neq 0$,
- (iii) $x \neq 0, y \neq 0$.

In the case (i) we have $f(x) = 0, f(y) = c$ and $\frac{xy}{x+y} = 0$, consequently $f\left(\frac{xy}{x+y}\right) = c$. The case (ii) is similar. In the case (iii) we have $f(x) = f(y) = 0$ and $\frac{xy}{x+y} \neq 0$, which give $f\left(\frac{xy}{x+y}\right) = 0$.

It seems natural to consider equation (E1) on the components of the domain of definition of rational operation (2). It turns out that equation (E1) considered on the components of the domain of definition of function (2) admits nonzero solutions which, however, have no more than two values. We proceed with a description of local solutions of (E1). We will describe non-trivial solutions on the components of the set

$$D = \{(x, y) \in \mathbb{R}^2 : x + y + \beta \neq 0\}$$

We will focus on the case $\beta > 0$. For $\beta < 0$ reasoning is similar. Namely, Theorem 3 for $\beta < 0$ may be proved similarly to Theorem 4 with $\beta > 0$ and analogously Theorem 4 for $\beta < 0$ may be proved similarly to Theorem 3 with $\beta > 0$. More precisely, if $\beta < 0$, $\beta = -\alpha$, $\alpha > 0$ and

$$f\left(\frac{xy}{x+y+\beta}\right) = f(x) + f(y), \quad x + y + \beta < 0,$$

then for g , given by $g(x) = f(-x)$, we have

$$g\left(\frac{xy}{x+y+\alpha}\right) = g(x) + g(y), \quad x + y + \alpha > 0.$$

THEOREM 3

The general local solution $f: \mathbb{R} \rightarrow \mathbb{R}$ of the functional equation

$$f\left(\frac{xy}{x+y+\beta}\right) = f(x) + f(y), \quad x + y + \beta > 0 \quad (\text{E1.2})$$

is

$$f(x) = \begin{cases} c & \text{for } x \leq -\beta, \\ 0 & \text{for } x > -\beta, \end{cases} \quad (5)$$

where $c \in \mathbb{R}$ is an arbitrary constant.

Proof. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a solution of (E1.2). Take $y = 0$ in (E1.2). Then

$$f(0) = f(x) + f(0), \quad x + \beta > 0.$$

Hence, we conclude that $f(x) = 0$ for every $x > -\beta$. By putting $x \leq -\beta$, $z = -x + \beta$ in (E1.2) we have $z > -\beta$ and $x + z = \beta > -\beta$ hence $f(z) = 0$ and therefore,

$$f\left(\frac{x(-x+\beta)}{2\beta}\right) = f(x)$$

for every $x \leq -\beta$. Observe that the implication

$$x \leq -\beta \rightarrow \frac{x(-x+\beta)}{2\beta} \leq -\beta$$

holds and therefore, we get $f(x) = f(y)$ for any $x, y \leq -\beta$, whence $f(x) = c$ for every $x \leq -\beta$, where $c \in \mathbb{R}$ is an arbitrary constant. Consequently f is of the form (5) where $c \in \mathbb{R}$ is arbitrary.

Obviously, each function of the form as above is a solution of (E1.2).

THEOREM 4

The general local solution $f: \mathbb{R} \rightarrow \mathbb{R}$ of the functional equation

$$f\left(\frac{xy}{x+y+\beta}\right) = f(x) + f(y), \quad x+y+\beta < 0 \quad (\text{E1.3})$$

is

$$f(x) = \begin{cases} 0 & \text{for } x < 0, \\ d & \text{for } x = 0, \\ c & \text{for } x > 0, \end{cases} \quad (6)$$

where $c, d \in \mathbb{R}$ are arbitrary constants.

Proof. Assume that $f: \mathbb{R} \rightarrow \mathbb{R}$ is a solution of (E1.3). By putting $x = y = -\beta$ in (E1.3) we have

$$f\left(\frac{\beta^2}{-\beta}\right) = f(-\beta) + f(-\beta).$$

Hence, we conclude that $f(-\beta) = 0$. By putting $x < 0$ and $y = -\beta$ in (E1.3) we get $x + y + \beta = x < 0$ and therefore,

$$f\left(\frac{-\beta x}{x}\right) = f(x)$$

for every $x < 0$. So we have $f(x) = 0$ for all $x < 0$.

Take an arbitrary $x > 0$. Then inequality $x + y + \beta < 0$ yields $y < 0$ and, therefore $\frac{xy}{x+y+\beta} > 0$. Consequently, by (E1.3) we obtain $f(x) = c$ for every $x > 0$. Thus the assertion follows.

It is not difficult to check that f given by (6) is a solution of (E1.3).

In order to obtain non trivial solutions of (E1) we exclude the lines $x = 0$, $y = 0$, $x = -\beta$, $y = -\beta$ from the set D where operation (2) is considered. We will describe solutions on these subsets of D which seem to be characteristic. Still, for the same reason as before, we will focus on the condition $\beta > 0$.

THEOREM 5

The general local solution $f: (0, +\infty) \rightarrow \mathbb{R}$ of the functional equation

$$f\left(\frac{xy}{x+y+\beta}\right) = f(x) + f(y), \quad x, y > 0 \quad (\text{E1.4})$$

is

$$f(x) = A \circ \ln \frac{x+\beta}{x},$$

where $A: (0, +\infty) \rightarrow \mathbb{R}$ is an arbitrary additive function.

Proof. First we observe, that the conditions $x > 0$, $y > 0$ imply

$$\frac{xy}{x+y+\beta} > 0.$$

Further, for homography $\varphi: (0, +\infty) \rightarrow (1, +\infty)$ given by

$$\varphi(x) = \frac{x + \beta}{x}$$

we have

$$\frac{xy}{x + y + \beta} = \varphi^{-1}(\varphi(x)\varphi(y))$$

for any $x, y > 0$. Let $f: (0, +\infty) \rightarrow \mathbb{R}$ be a solution of equation (E1.4). For every $x > 0, y > 0$ we may substitute $\varphi(x) = u > 1, \varphi(y) = v > 1$ in (E1.4) to obtain

$$g(uv) = g(u) + g(v), \quad u, v > 1 \quad (7)$$

for the function $g = f \circ \varphi^{-1}$. Further, for every $u > 1$ and $v > 1$ we may substitute $u = e^s, v = e^t$ in (7) to get

$$A(s + t) = A(s) + A(t), \quad s, t > 0$$

for the function $A: (0, +\infty) \rightarrow \mathbb{R}$ given by $A(x) = g(e^x)$. This implies $f = A \circ \ln \circ \varphi$, which ends the proof.

Because in the above proof the function $g: (1, +\infty) \rightarrow \mathbb{R}$ is exponential, writing H instead of g we obtain

REMARK

The general local solution $f: (0, +\infty) \rightarrow \mathbb{R}$ of the functional equation (E1.4) can be written in the form

$$f(x) = H\left(\frac{x + \beta}{x}\right),$$

where $H: (1, +\infty) \rightarrow \mathbb{R}$ is an arbitrary continuous exponential function.

Analogously we can prove the following result.

THEOREM 6

The general local solution $f: (-\infty, -\beta) \rightarrow \mathbb{R}$ of the functional equation

$$f\left(\frac{xy}{x + y + \beta}\right) = f(x) + f(y), \quad x, y < -\beta$$

is

$$f(x) = A \circ \ln \frac{x + \beta}{x},$$

where $A: (-\infty, 0) \rightarrow \mathbb{R}$ is an arbitrary additive function.

THEOREM 7

If the function $f: (-\infty, 0) \setminus \{-\beta\} \rightarrow \mathbb{R}$ is a solution of the functional equation

$$f\left(\frac{xy}{x + y + \beta}\right) = f(x) + f(y), \quad x \in (-\beta, 0), y \in (-\infty, -\beta) \quad (\text{E1.6})$$

then there exist an additive function $A: \mathbb{R} \rightarrow \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that

$$f(x) = \begin{cases} A \circ \ln \frac{x + \beta}{x} & \text{for } x \in (-\infty, -\beta), \\ A \circ \ln\left(-\frac{x + \beta}{x}\right) + c & \text{for } x \in (-\beta, 0). \end{cases} \quad (8)$$

Conversely, for any additive function $A: \mathbb{R} \rightarrow \mathbb{R}$ and any constant $c \in \mathbb{R}$ the function f given by (8) is a solution of equation (E1.6).

Proof. First we observe, that from $x \in (-\beta, 0)$ and $y \in (-\infty, -\beta)$ follows that

$$\frac{xy}{x+y+\beta} \in (-\beta, 0).$$

Further, for homography $\varphi: (-\infty, 0) \setminus \{-\beta\} \rightarrow (-\infty, 1) \setminus \{0\}$ given by

$$\varphi(x) = \frac{x+\beta}{x},$$

we have

$$\frac{xy}{x+y+\beta} = \varphi^{-1}(\varphi(x)\varphi(y))$$

for every $x \in (-\beta, 0)$ and $y \in (-\infty, -\beta)$. Assume that $f: (-\infty, 0) \setminus \{-\beta\} \rightarrow \mathbb{R}$ is a solution of (E1.6). For $x \in (-\beta, 0)$, $y \in (-\infty, -\beta)$ we may substitute $\varphi(x) = u < 0$ and $\varphi(y) = v \in (0, 1)$ in (E1.6) to obtain

$$g(uv) = g(u) + g(v), \quad u < 0, v \in (0, 1) \quad (9)$$

for the function $g = f \circ \varphi^{-1}$. Further, for any $u \in (-\infty, 0)$, $v \in (0, 1)$ substitution $u = -e^t$, $v = e^s$ in (9) yields

$$h(s+t) = h(t) + k(s), \quad t \in \mathbb{R}, s < 0 \quad (10)$$

for functions $h: \mathbb{R} \rightarrow \mathbb{R}$ and $k: (-\infty, 0) \rightarrow \mathbb{R}$ given by $h(x) = g(-e^x)$ and $k(x) = g(e^x)$. For an arbitrary fixed $t_0 \in \mathbb{R}$ by putting $t = t_0$ in (10) we obtain

$$k(s) = h(s+t_0) - c_1, \quad s < 0,$$

where $c_1 = h(t_0)$. Now we have by (10),

$$a(s+t) = a(s) + a(t), \quad t \in \mathbb{R}, s \in (-\infty, 0),$$

where $a: \mathbb{R} \rightarrow \mathbb{R}$ is given by $a(t) = h(t+t_0) - c_1$. Hence the function $A: \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$A(t) = \begin{cases} a(t) & \text{for } t < 0, \\ 0 & \text{for } t = 0, \\ -a(-t) & \text{for } t > 0 \end{cases}$$

is additive. Because $h(t) = A(t-t_0) + c_1 = A(t) + c$, $t \in \mathbb{R}$, where $c = c_1 - A(t_0)$ and $k(s) = A(s)$, $s < 0$ we obtain

$$g(x) = \begin{cases} A \circ \ln(-x) + c & \text{for } x < 0, \\ A \circ \ln x & \text{for } x \in (0, 1). \end{cases}$$

Hence

$$f(x) = \begin{cases} A \circ \ln \circ (-\varphi)(x) + c & \text{for } \varphi(x) < 0, \\ A \circ \ln \circ \varphi(x) & \text{for } \varphi(x) \in (0, 1), \end{cases}$$

which implies (8).

It is easy to check that (8) is a solution of (E1.6). Thus, the proof is completed.

Finally we prove

THEOREM 8

If the function $f: (-\infty, -\beta) \cup (0, +\infty) \rightarrow \mathbb{R}$ is a solution of the functional equation

$$f\left(\frac{xy}{x+y+\beta}\right) = f(x) + f(y), \quad x > 0, y < -\beta, x+y+\beta < 0 \quad (\text{E1.7})$$

then there exist an additive function $A: \mathbb{R} \rightarrow \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that

$$f(x) = \begin{cases} A \circ \ln \frac{x+\beta}{x} & \text{for } x \in (-\infty, -\beta), \\ A \circ \ln \frac{x+\beta}{x} + c & \text{for } x \in (0, +\infty). \end{cases} \quad (11)$$

Conversely, for any additive function $A: \mathbb{R} \rightarrow \mathbb{R}$ and a constant $c \in \mathbb{R}$ the function f given by (11) satisfies equation (E1.7).

Proof. First we observe, that for $x \in (0, +\infty)$, $y \in (-\infty, -\beta)$ and $x+y+\beta < 0$ the following inequality holds

$$\frac{xy}{x+y+\beta} > 0.$$

Further, for homography $\varphi: (-\infty, -\beta) \cup (0, +\infty) \rightarrow (0, 1) \cup (1, +\infty)$, given by the formula

$$\varphi(x) = \frac{x+\beta}{x},$$

we have

$$\frac{xy}{x+y+\beta} = \varphi^{-1}(\varphi(x)\varphi(y))$$

for every $x \in (0, +\infty)$ and $y \in (-\infty, -\beta)$. Let $f: (-\infty, -\beta) \cup (0, +\infty) \rightarrow \mathbb{R}$ be a solution of (E1.7). Then

$$f \circ \varphi^{-1}(\varphi(x)\varphi(y)) = f \circ \varphi^{-1}(\varphi(x)) + f \circ \varphi^{-1}(\varphi(y))$$

for $x \in (0, +\infty)$, $y \in (-\infty, -\beta)$, $x+y+\beta < 0$ and, putting here $u = \varphi(x)$, $v = \varphi(y)$, we obtain

$$g(uv) = g(u) + g(v), \quad u > 1, v \in (0, 1), uv > 1 \quad (12)$$

for the function $g = f \circ \varphi^{-1}: (0, +\infty) \setminus \{1\} \rightarrow \mathbb{R}$. Further, for any $u \in (1, +\infty)$, $v \in (0, 1)$ we may substitute $u = e^t$ and $v = e^s$ in (12) to obtain

$$h(s+t) = h(t) + h(s), \quad t > 0, s < 0, t+s > 0$$

for the function $h: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ given by $h(x) = g(e^x)$. Applying here the well-know extension theorem of Z. Daróczy and L. Losonczi [2] we conclude that there exists an additive function $A: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\begin{aligned} A(t) - A(u) - A(v) &= h(t) - h(u) - h(v), & t > 0, \\ A(t) - A(u) &= h(t) - h(u), & t > 0, \end{aligned}$$

$$A(t) - A(v) = h(t) - h(v), \quad t < 0,$$

where $u > 0$, $v < 0$ and $u + v > 0$. The first two equalities give $A(v) = h(v)$ for all $v < 0$. For an arbitrary fixed $u_0 > 0$ the second equality implies also

$$h(t) = A(t) - A(u_0) + h(u_0), \quad t > 0.$$

Therefore,

$$h(t) = \begin{cases} A(t) & \text{for } t < 0, \\ A(t) + c & \text{for } t > 0, \end{cases}$$

where $c := h(u_0) - A(u_0)$. It follows that for any $x \in (0, +\infty) \setminus \{1\}$,

$$g(x) = \begin{cases} A \circ \ln x & \text{for } x \in (0, 1), \\ A \circ \ln x + c & \text{for } x \in (1, +\infty), \end{cases}$$

which yields (12).

Obviously each function given by (12) satisfies (E1.7). This ends the proof.

We will end the description of the solutions of (E1) with a theorem, the proof of which is similar to the proof of Theorem 8.

THEOREM 9

If the function $f: (-\infty, -\beta) \cup (0, +\infty) \rightarrow \mathbb{R}$ is a solution of the functional equation

$$f\left(\frac{xy}{x+y+\beta}\right) = f(x) + f(y), \quad x > 0, y < -\beta, x+y+\beta > 0, \quad (\text{E1.8})$$

then there exist an additive function $A: \mathbb{R} \rightarrow \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that

$$f(x) = \begin{cases} A \circ \ln \frac{x+\beta}{x} + c & \text{for } x \in (-\infty, -\beta), \\ A \circ \ln \frac{x+\beta}{x} & \text{for } x \in (0, +\infty). \end{cases}$$

Conversely, for any additive function $A: \mathbb{R} \rightarrow \mathbb{R}$ and any constant $c \in \mathbb{R}$ the above function f satisfies (E1.8).

Now we consider the functional equation (E2). For $\alpha = 0$ it is equivalent to (E1) with $\beta = 0$. L. Losonczi in [4] determined the local solutions of this equation for $\alpha = 1$ and for $\alpha = -1$. Among others he proved, that

THEOREM 10

(i) The general local solution function of the functional equation

$$f\left(\frac{xy-1}{x+y}\right) = f(x) + f(y), \quad x+y > 0$$

is $f: \mathbb{R} \rightarrow \mathbb{R}$ of the form $f(x) = A_1(\operatorname{arccot} x)$.

(ii) The general local solution function of the functional equation

$$f\left(\frac{xy-1}{x+y}\right) = f(x) + f(y), \quad x+y < 0$$

is $f: \mathbb{R} \rightarrow \mathbb{R}$ of the form $f(x) = A_2(\operatorname{arccot} x - \pi)$.

(iii) The general local solution function of the functional equation

$$f\left(\frac{xy+1}{x+y}\right) = f(x) + f(y), \quad x, y \in (-1, 1), \quad x+y \neq 0$$

is $f: \mathbb{R} \setminus \{-1, 1\} \rightarrow \mathbb{R}$ of the form

$$f(x) = \begin{cases} A_3(\operatorname{arcoth} x) + 2b & \text{for } |x| > 1, \\ A_4(\operatorname{artanh} x) + b & \text{for } |x| < 1. \end{cases}$$

(iv) The general local solution function of the functional equation

$$f\left(\frac{xy+1}{x+y}\right) = f(x) + f(y), \quad x < -1, \quad y > 1, \quad x+y \neq 0$$

is $f: \mathbb{R} \setminus [-1, 1] \rightarrow \mathbb{R}$ of the form $f(x) = A_5(\operatorname{arcoth} x)$.

$A_i: \mathbb{R} \rightarrow \mathbb{R}$, $i \in \{1, 2, 3, 4, 5\}$ are arbitrary additive functions and $b \in \mathbb{R}$ is an arbitrary constant.

Cases (iii) and (iv) are not explicitly written in his paper [4]. Furthermore noticing that

$$\begin{aligned} \operatorname{artanh} x &= \frac{1}{2} \ln \frac{1+x}{1-x}, & x \in (-1, 1), \\ \operatorname{arcoth} x &= \frac{1}{2} \ln \frac{x+1}{x-1}, & x \in \mathbb{R} \setminus [-1, 1], \end{aligned}$$

we can write in (iii),

$$f(x) = \begin{cases} A_6 \circ \ln \frac{x+1}{x-1} + 2b & \text{for } |x| > 1, \\ A_7 \circ \ln \frac{1+x}{1-x} + b & \text{for } |x| < 1 \end{cases}$$

and

$$f(x) = A_8 \circ \ln \frac{x+1}{x-1}$$

in (iv) with arbitrary additive functions $A_i: \mathbb{R} \rightarrow \mathbb{R}$, $i \in \{6, 7, 8\}$. It turned out that the solutions of equation (E2) on the right sets can be described by the solutions of equations from the above theorem. It is not difficult to prove that

THEOREM 11

- (i) The function f is a solution of equation (E2) in the case $\alpha < 0$ if and only if $f = g \circ \varphi$, where $\varphi(x) = \frac{x}{\sqrt{-\alpha}}$, $x \in \mathbb{R}$ and g is a solution of (E2) with $\alpha = -1$.
- (ii) The function f is a solution of equation (E2) in the case $\alpha > 0$ if and only if $f = g \circ \varphi$, where $\varphi(x) = \frac{x}{\sqrt{\alpha}}$, $x \in \mathbb{R}$ and g is a solution of (E2) with $\alpha = 1$.

Proof. For the proof it is enough to notice that

$$\frac{xy + \alpha}{x + y} = \varphi^{-1} \left(\frac{\varphi(x)\varphi(y) + 1}{\varphi(x) + \varphi(y)} \right),$$

where $\varphi(x) = \frac{x}{\sqrt{\alpha}}$ if $\alpha > 0$ and

$$\frac{xy + \alpha}{x + y} = \varphi^{-1} \left(\frac{\varphi(x)\varphi(y) - 1}{\varphi(x) + \varphi(y)} \right),$$

where $\varphi(x) = \frac{x}{\sqrt{-\alpha}}$ if $\alpha < 0$.

As the consequence, we get local solutions of equation (E2).

THEOREM 12

(i) *The general local solution function of the functional equation*

$$f \left(\frac{xy + \alpha}{x + y} \right) = f(x) + f(y), \quad x + y > 0,$$

where $\alpha < 0$ is $f: \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$f(x) = A_1 \left(\operatorname{arccot} \frac{x}{\sqrt{-\alpha}} \right).$$

(ii) *The general local solution function of the functional equation*

$$f \left(\frac{xy + \alpha}{x + y} \right) = f(x) + f(y), \quad x + y < 0,$$

where $\alpha < 0$ is $f: \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$f(x) = A_2 \left(\operatorname{arccot} \frac{x}{\sqrt{-\alpha}} - \pi \right).$$

(iii) *The general local solution function of the functional equation*

$$f \left(\frac{xy + \alpha}{x + y} \right) = f(x) + f(y), \quad x, y \in (-\sqrt{\alpha}, \sqrt{\alpha}), \quad x + y \neq 0,$$

where $\alpha > 0$ is $f: \mathbb{R} \setminus \{-\sqrt{\alpha}, \sqrt{\alpha}\} \rightarrow \mathbb{R}$ of the form

$$f(x) = \begin{cases} A_3 \left(\operatorname{arccoth} \frac{x}{\sqrt{\alpha}} \right) + 2b & \text{for } |x| > \sqrt{\alpha}, \\ A_4 \left(\operatorname{artanh} \frac{x}{\sqrt{\alpha}} \right) + b & \text{for } |x| < \sqrt{\alpha}. \end{cases}$$

(iv) *The general local solution function of the functional equation*

$$f \left(\frac{xy + \alpha}{x + y} \right) = f(x) + f(y), \quad x < -1, \quad y > 1, \quad x + y \neq 0,$$

where $\alpha > 0$ is $f: \mathbb{R} \setminus [-\sqrt{\alpha}, \sqrt{\alpha}] \rightarrow \mathbb{R}$ of form

$$f(x) = A_5 \left(\operatorname{arccoth} \frac{x}{\sqrt{\alpha}} \right).$$

$A_i: \mathbb{R} \rightarrow \mathbb{R}$, $i \in \{1, 2, 3, 4, 5\}$ are arbitrary additive functions and $b \in \mathbb{R}$ is an arbitrary constant.

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