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Report of Meeting

18th International Conference on Functional Equations and Inequalities, Będlewo, Poland, June 9–15, 2019

The **18th International Conference on Functional Equations and Inequalities** (18th ICFEI) took place in the *Mathematical Research and Conference Center* in Będlewo (Poland) on June 9–15, 2019. It was organized by the Department of Mathematics of the Pedagogical University of Kraków.

The Scientific Committee of the 18th ICFEI consisted of Professors: Nicole Brillouët-Belluot (France), Janusz Brzdęk (Poland) – chairman, Jacek Chmieliński (Poland), Roman Ger (Poland), Zsolt Páles (Hungary), Dorian Popa (Romania), Ekaterina Shulman (Poland/Russia), Henrik Stetkær (Denmark), László Székelyhidi (Hungary) and Marek Cezary Zdun (Poland).

The Organizing Committee consisted of Jacek Chmieliński (chairman), Zbigniew Leśniak (vice-chairman), Beata Deręgowska (managing secretary), Paweł Pasteczka (scientific secretary), Paweł Wójcik (scientific secretary) and Paweł Solarz (web & technical support).

There were 56 participants who came from 15 countries: Austria (2 participants), China (2), Denmark (1), Finland (1), France (1), Germany (2), Hungary (6), India (1), Israel (1), Morocco (1), Poland (30), Romania (5), Serbia (1), South Africa (1), USA (1).

The conference was opened on Monday, June 10, by Professors Janusz Brzdęk, the Chairman of the Scientific Committee, and Jacek Chmieliński, the Chairman of the Organizing Committee. The opening ceremony was followed by the plenary lecture of Professor Roman Ger.

During 20 scientific sessions 44 talks were presented, including longer plenary lectures delivered by Professors Roman Ger, Justyna Sikorska, Kazimierz Nikodem and Eliza Jabłońska. The talks were devoted mainly to functional equations and inequalities, convexity, stability of functional equations, means, as well as to related topics, in particular in real and functional analysis, and applications. Additionally, apart from regular talks, *Problems and Remarks* sessions were scheduled.

Several social events accompanied the conference. On Wednesday afternoon participants could enjoy an excursion to the Museum of First Piasts at Lednica, including the archaeological reservation at Ostrów Lednicki (Lednicki Island), and to the Wielkopolski Ethnography Park. A picnic with a bonfire was organized on Tuesday evening and a banquet on Thursday. On Friday evening a piano concert was performed by Dr. Marek Czerni.

The conference was closed on Saturday, June 15, by Professor Janusz Brzdęk. As it was announced, the subsequent 19th ICFEI will be organized on September 12–18, 2021, again in Będlewo.

1. Abstracts of Talks

Mirosław Adamek *On Hermite-Hadamard type inequalities for F-convex functions*

Let I be a nonempty and open interval of $\mathbb{R}$ and $F\colon \mathbb{R}\to\mathbb{R}$ be a fixed function. A function $f\colon I\to\mathbb{R}$ will be called *F-convex* if

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)-t(1-t)F(x-y)$$

for all $x,y\in I$ and $t\in(0,1)$.

The classical Hermite-Hadamard inequality says that for any convex function $f\colon I\to\mathbb{R}$ we have

$$f\Big(\frac{x+y}{2}\Big)\le\frac{1}{y-x}\int_x^y f(u)du\le\frac{f(x)+f(y)}{2}$$

for all different $x,y\in I$.

In this talk we present Hermite-Hadamard type inequalities for F-convex functions. As a consequence of our investigations we get the following inequality

$$\begin{aligned}f\Big(\frac{x+y}{2}\Big)+\frac{1}{y-x}\int_x^y F\Big(u-\frac{x+y}{2}\Big)du&\le\frac{1}{y-x}\int_x^y f(u)\,du\\&\le\frac{f(x)+f(y)}{2}-\frac{1}{6}F(x-y)\end{aligned}$$

for all different $x,y\in I$.

Nutefe Kwami Agbeko *On some lattice-valued functional equations and inequalities*

Since early 90's we have considered lattice-valued functions and operators defined on diverse sets with algebraic structures, by replacing the addition with lattice operations. In this perspective, the addition in the definition of the probability measure as well as in the definition of the Lebesgue's integral (or mathematical expectation) is substituted with the supremum operation and the so-defined functions (*optimal measure*, resp. *optimal average*) behave similarly like their counterparts in Measure Theory. From 2012 the Cauchy functional equation has also been studied for lattice-valued functions (which we can term *lattice-valued Cauchy functional equation*), where the supremum replaces the addition in the Cauchy functional equation, focusing on the Ulam's type stability. The main goal of our presentation is to talk about morphisms between sets with an algebraic structure and an order structure, through associated functional equations and inequalities: we discuss the separation problem for the inequalities and the Hyers-Ulam stability of the main equation.

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Pekka Alestalo *Sharp extension results for bilipschitz maps*

An L-bilipschitz map $f\colon A \to \mathbf{R}^n$ satisfies the double inequality

$$\|x-y\|/L \leq \|f(x)-f(y)\| \leq L\|x-y\|$$

for all $x, y \in A \subset \mathbf{R}^n$. For topological reasons, these maps cannot usually be extended to a homeomorphism $F\colon \mathbf{R}^n \to \mathbf{R}^n$. In the other extreme, it is sometimes possible to find an L-bilipschitz extension F with the same constant L. We present some positive results and examples related to this problem.

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Alina Ramona Baias *On the best Ulam constant of a third order linear difference equation*

Let X be a Banach space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We give a result on Ulam stability for the linear difference equation

$$x_{n+3} = ax_{n+2} + bx_{n+1} + cx_n, \quad n \geq 0, \tag{1}$$

where $a, b, c \in \mathbb{K}$, $x_0, x_1, x_2 \in X$. Moreover, if all the roots of the characteristic equation of (1) have the modulus greater then 1, we obtain the best Ulam constant of the equation.

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Karol Baron *Remarks on continuous solutions of an iterative functional equation*

Assume X is a real separable Hilbert space, $\Lambda\colon X \to X$ is linear and continuous with $\|\Lambda\| < 1$, and μ is a probability Borel measure on X with finite first moment. We examine continuous at zero solutions $\varphi\colon X \to \mathbb{C}$ of the equation

$$\varphi(x) = \hat{\mu}(x)\varphi(\Lambda x).$$

Liviu Cădariu-Brăiloiu *Generalized Hyers-Ulam stability of some functional equations*

The aim of this talk is to present several generalized Hyers-Ulam stability properties for some functional equations, by using the fixed point method.

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Jacek Chudziak *Convexity and quasi-convexity of the zero utility principle*

Assume that $\mathcal{X}_+$ is a family of all nonnegative bounded random variables on a given probability space. The elements of $\mathcal{X}_+$ represent the risks to be insured by an insurance company. An important question of the theory of insurance risk premiums is to assign to every $X \in \mathcal{X}_+$ a premium for the insurance contract. One of the methods of determining the premium is *the zero utility principle*. It defines the premium for a risk $X \in \mathcal{X}_+$ as a real number $H_u(X)$ satisfying equation

$$E[u(H_u(X) - X)] = 0 \tag{1}$$

where $u\colon \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function with $u(0) = 0$. It turns out that, for every $X \in \mathcal{X}_+$, such a number $H_u(X)$ exists and it is unique. Therefore equation (1) determines in an implicit way a functional $H_u\colon \mathcal{X}_+ \to \mathbb{R}$.

Several results concerning properties of the functional H_u can be found, e.g. in [1, 2, 3] and [6]. It is known that, for every strictly increasing continuous function $u : \mathbb{R} \to \mathbb{R}$ satisfying $u(0) = 0$, H_u is monotone and conditionally translation invariant. However, in general, it is not convex. In this talk we present a characterization of convexity and quasi-convexity of H_u. A fundamental role in our investigations is played by *quasideviation means* (cf. [4, 5]).

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Włodzimierz Fechner *Sincov's inequalities on topological spaces*

During the talk we will discuss the multiplicative Sincov's inequality

$$G(a,b) \le G(a,x) \cdot G(x,b), \quad a,x,b \in X.$$

We assume that X is a topological space and G is a continuous map. We also study the reverse inequality

$$F(a,b) \ge F(a,x) \cdot F(x,b), \quad a,x,b \in X$$

and the additive version of the original inequality

$$H(x,z) \le H(x,y) + H(y,z), \quad x,y,z \in X.$$

A corollary for generalized metric is derived.

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Roman Ger *On convex type functional inequalities*

The lecture focuses on some weak regularity requirements forcing the automatic continuity of real convex functionals on normed spaces and, more generally, on locally convex linear topological spaces, supporting and separation theorems (i.e. geometric counterpats of various generalizations of the celebrated Hahn-Banach extension theorem). In some situations, groups will be considered as potential domains and the related problem of their classification with respect to the question whether or not they admit invariant means (amenability).

Because of the significant role and good geometry of strictly convex spaces some characterizations of them in terms of solutions of the fundamental Cauchy functional equation assumed to be satisfied *modulo* a norm with the emphasis given to the factorization of these solutions into the additive and isometric parts. Several further necessary and sufficient conditions for the strict convexity of the given space will be analyzed as well along with some characterizations of inner product spaces with the aid of suitable iterations of the difference operators.

Delta-convex mappings between normed linear spaces provide a generalization of functions which are representable as a difference of two convex functions to the case of vector valued maps. Following L. Veselý and L. Zajíček, (*Delta-convex mappings between Banach spaces and applications,* Dissertationes Math. 289, Polish Scientific Publishers, Warszawa, 1989) we show that this class of mappings has very good properties proving that a generalization proposed is well established. Strict connections with the Hyers-Ulam stability in the theory of functional equations and inequalities will be revealed. We look also for possibly mild regularity conditions upon the maps whose vector convex differences are controlled by their scalar counterparts, forcing these maps to be delta-convex. Finally, vector analogues of the celebrated Hermite-Hadamard type inequalities will also be presented.

Moshe Goldberg *Extending the spectral radius to finite-dimensional power-associative algebras*

The purpose of this talk is to introduce a new concept, the *radius* of elements in arbitrary finite-dimensional power-associative algebras over the field of real or complex numbers. It is an extension of the well known notion of the spectral radius.

As examples, we shall discuss this new radius in the setting of matrix algebras, where it indeed reduces to the spectral radius, and then in the Cayley-Dickson algebras, where it is something quite different.

Richárd Grünwald *Characterization of the equality of generalized Bajraktarević means*

The purpose of the talk is to investigate the equality problem of generalized Bajraktarević means, i.e. to solve the functional equation

$$f^{(-1)}\left(\tfrac{p_1(x_1)f(x_1)+\dots+p_n(x_n)f(x_n)}{p_1(x_1)+\dots+p_n(x_n)}\right) = g^{(-1)}\left(\tfrac{q_1(x_1)g(x_1)+\dots+q_n(x_n)g(x_n)}{q_1(x_1)+\dots+q_n(x_n)}\right), \qquad (*)$$

which holds for all $x = (x_1, \dots, x_n) \in I^n$, where $n \geq 2$, I is a nonempty open real interval, the unknown functions $f, g\colon I \to \mathbb{R}$ are strictly monotone, $f^{(-1)}$ and $g^{(-1)}$ denote their generalized left inverses, respectively, and $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$ and $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}_+^n$ are also unknown functions. This equality problem in the symmetric two-variable (i.e. when $n = 2$) case was already investigated and solved under sixth-order regularity assumptions by Losonczi in 1999. In the nonsymmetric two-variable case, assuming three times differentiability of f, g and the existence of $i \in \{1, 2\}$ such that either p_i is twice continuously differentiable and p_{3-i} is continuous on I, or p_i is twice differentiable and p_{3-i} is once differentiable on I, we prove that (*) holds if and only if there exist four constants $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that

$$cf + d > 0, \quad g = \frac{af + b}{cf + d} \quad \text{and} \quad q_\ell = (cf + d)p_\ell \qquad (\ell \in \{1, \dots, n\}).$$

In the case $n \geq 3$, we obtain the same conclusion with weaker regularity assumptions. Namely, we suppose that f and g are three times differentiable, p is continuous and there exist $i, j, k \in \{1, \dots, n\}$ with $i \neq j \neq k \neq i$ such that p_i, p_j, p_k are differentiable.

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László Horváth *Sharp Gronwall-Bellman type integral inequalities with delay*

Various attempts have been made to give an upper bound for the solutions of the delayed version of the Gronwall-Bellman integral inequality, but the obtained estimations are not sharp. In this talk a new approach is presented to get sharp estimations for the nonnegative solutions of the considered delayed inequalities. The results are based on the idea of the generalized characteristic inequality. Our method gives sharp estimation, and therefore the results are more exact than the earlier ones.

Eliza Jabłońska *Haar 'small' sets in abelian Polish groups*

It is well known [1] that a subset A of an abelian Polish group X is called *Haar null* if there are a universally measurable set $B \subset X$ with $A \subset B$ and a Borel probability measure μ on X such that $\mu(x + B) = 0$ for all $x \in X$. In [2] Darji introduced another family of 'small' sets in an abelian Polish group X;

he called a set $A \subset X$ *Haar meager* if there is a Borel set $B \subset X$ with $A \subset B$, a compact metric space K and a continuous function $f\colon K \to X$ such that

$$f^{-1}(B+x) \text{ is meager in } K \text{ for every } x \in X.$$

In a locally compact group these two definitions are equivalent to definitions of Haar measure zero sets and meager sets, respectively. That is why we can say that the notion of a Haar meager set is a topological analog to the notion of a Haar null set. Since lots of similarities between meager sets and sets of Haar measure zero are well known in locally compact groups (see e.g. [3]), we would like to present some analogies between Haar meager sets and Haar null sets.

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Zbigniew Leśniak *On fixed points of functions with values in a dq-metric space* (joint work with Janusz Brzdęk and El-sayed El-hady)

We present a fixed-point theorem for an operator acting on some classes of functions with values in a dq-metric space and show its applications to prove the stability in Ulam sense of some types of functional and difference equations. The fixed points of such operators turn out to be exact solutions of the considered equations that meet the imposed conditions.

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Renata Malejki *On Ulam stability of a generalization of the Fréchet functional equation on a restricted domain*

In this paper we prove the Ulam type stability of a generalization of the Fréchet functional equation on a restricted domain. In the proofs the main tool is a fixed point theorem for some function spaces.

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Janusz Matkowski *Quasi-Cauchy difference means*

Quasi-Cauchy difference means of:

(i) additive type, i.e. the functions of the form

$$M_f(x_1,\dots,x_k)=F^{-1}(f(x_1+...+x_k)-(f(x_1)+\cdots+f(x_k))),$$

where $F(x)=f(kx)-kf(x)$;

(ii) exponential type, i.e. the functions of the form

$$E_f(x_1,\dots,x_k)=F^{-1}\Big(f\Big(\sum_{j=1}^k x_j\Big)-\prod_{j=1}^k f(x_j)\Big),$$

where $F(x)=f(kx)-[f(x)]^k$;

(iii) logarithmic type, i.e. the functions of the form

$$L_f(x_1,\dots,x_k)=F^{-1}\Big(f\Big(\prod_{j=1}^k x_j\Big)-\sum_{j=1}^k f(x_j)\Big)$$

where $F(x):=f(x^k)-kf(x)$;

(iv) of power type, i.e. the functions of the form

$$P_f(x_1,\dots,x_k)=F^{-1}\Big(f\Big(\prod_{j=1}^k x_j\Big)-\prod_{j=1}^k f(x_j)\Big),$$

where $F(x):=f(x^k)-[f(x)]^k$,

as well as the respective functions with difference replaced by division, will be considered.

Janusz Morawiec *Around a Kazimierz Nikodem result - part I*
(joint work with Thomas Zürcher)

Let $(X,\mathcal{A},\mu)$ be a probability space and let $S\colon X\to X$ be a measurable transformation. Motivated by the paper of K. Nikodem [1], we concentrate on a functional equation generating measures that are absolutely continuous with respect to μ and ε-invariant under S. As a consequence of the investigation, we obtain a result on the existence and uniqueness of solutions $\varphi\in L^1([0,1])$ of the functional equation

$$\varphi(x)=\sum_{n=1}^N |f_n'(x)|\varphi(f_n(x))+g(x),$$

where $g\in L^1([0,1])$ and $f_1,\dots,f_N\colon[0,1]\to[0,1]$ are functions satisfying some extra conditions. The results we are going to present were recently published in [2].

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Jacek Mrowiec *On strongly convex functions of higher order*

Two given finite sequences $(b_1, \ldots, b_m)$ and $(c_1, \ldots, c_m)$, where $m \in \mathbb{N}$, $b_k \in \{0, 1\}$, $c_k > 0$, $k = 1, \ldots, m$, describe a function F defined on a bounded interval I in the following way:

$b_k = 1$ means that F is strongly convex of order k with modulus c_k on I,

$b_k = 0$ means that F is strongly concave of order k with modulus c_k on I,

$k = 1, \ldots, m$.

For any fixed pair of such sequences the question of existence of F arises. In the talk the construction of the example of a function with the desired property will be presented. The case of infinite sequences also will be considered.

Kazimierz Nikodem *On strongly convex functions and related classes of functions*

Let D be a convex subset of a normed space and $c > 0$. A function $f \colon D \to \mathbb{R}$ is called *strongly convex* with modulus c if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2$$

for all $x, y \in D$ and $t \in [0, 1]$; f is called *strongly midconvex* with modulus c if

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{f(x) + f(y)}{2} - \frac{c}{4}\|x - y\|^2, \quad x, y \in D.$$

Strongly convex functions are useful in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature. In my talk some results on strongly convex functions and related classes of functions obtained by the author with co-authors in the last few years are presented. In particular, discrete and integral Jensen-type inequalities and a Hermite-Hadamard-type inequality for strongly convex functions are obtained. Counterparts of the classical Bernstein-Doetsch and Sierpiński theorems for strongly midconvex functions are given. New characterizations of inner product spaces involving strong convexity are obtained. A representation of strongly Wright-convex functions and a characterization of functions generating strongly Schur-convex sums are presented. Finally, some properties of strongly convex set-valued maps and strongly convex stochastic processes are presented.

Diana Otrocol *Functional equations and entropies*

We consider entropies corresponding to some probability distributions and establish functional/differential equations satisfied by them. Connections between these entropies are studied. As applications we investigate shape properties of the entropies and derive combinatorial identities.

Lahbib Oubbi *Hyers-Ulam stability and hyperstability of a general functional equation in random normed spaces, a purely fixed point approach*

If X is a real or complex vector space, (Y, F, T_M) is a complete Random normed space, and $f\colon X \to Y$ a mapping, then using a purely fixed point approach, we prove the Ulam-Hyers stability and hyperstability of the general functional equation

$$\sum_{i=1}^{m} A_i f\Big(\sum_{j=1}^{n} a_{ij} x_j\Big) + A = 0.$$

Here f is a mapping from X into a Random normed space (Y, F, T_M), m and n are positive integers, for every $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, A_i and a_{ij} are scalars, and A is a vector from Y. Several known results can be derived.

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Zsolt Páles *Optimal error functions for approximately monotone and convex functions*

Let I be a nonempty open real interval and let $\ell(I) \in]0, \infty]$ denote its length. Given a nonnegative error function $\Phi\colon [0, \ell(I)[\to \mathbb{R}_+$, a function $f\colon I \to \mathbb{R}$ will be called a Φ-*monotone* function if, for all $x, y \in I$ with $x \leq y$,

$$f(x) \leq f(y) + \Phi(y - x).$$

We say that a function $f\colon I \to \mathbb{R}$ is Φ-*convex* if, for all $x, y \in I$ and $t \in [0, 1]$, it satisfies the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t\Phi((1-t)|x-y|) + (1-t)\Phi((t|x-y|).$$

In the talk, we discuss the following problem. If $\Phi\colon [0, \ell(I)[\to \mathbb{R}_+$ is an error function then determine the smallest error function $\Phi^*\colon [0, \ell(I)[\to \mathbb{R}_+$ such that Φ-monotonicity and Φ-convexity imply Φ^*-monotonicity and Φ^*-convexity, respectively.

Rajendra Pant *Viscosity approximation methods for multi-valued nonexpansive mappings*

We present some viscosity approximation theorems for multi-valued generalized nonexpansive mappings with applications to variational inequality and split common fixed point problems. Some numerical computations will be presented to illustrate our results.

Paweł Pasteczka *Weakening of Hardy property for means*

The aim of this talk is to find a broad family of means defined on a subinterval of $I \subset [0, +\infty)$ such that

$$a_1 + \mathscr{M}(a_1, a_2) + \mathscr{M}(a_1, a_2, a_3) + \cdots < +\infty \quad \text{for all } a \in \ell_1(I).$$

Equivalently, the averaging operator

$$(a_1,\, a_2, a_3\,, \dots) \mapsto (a_1,\, \mathscr{M}(a_1, a_2),\, \mathscr{M}(a_1, a_2, a_3), \dots)$$

is a selfmapping of $\ell_1(I)$. This property is closely related to so-called Hardy inequality for means (which additionally requires boundedness of this operator).

We prove that these two properties are equivalent in a broad family of Gini means. Moreover, it is shown that this is not the case for quasi-arithmetic means. However, weak-Hardy property is localizabile for this family.

Dorian Popa *Ulam stability of an operatorial difference equation*

Let T be a bounded linear operator acting on a Banach space X. We obtain some results on Ulam stability for the linear difference equation $x_{n+1} = Tx_n + a_n$ associated to an iterative process for the linear equation $x - Tx = y$. As applications we get some stability results for the case when X is a finite dimensional space and for the case when T is Fredholm operator.

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Teresa Rajba *On some inequalities for Bernstein operators and convex functions*

For $n \in \mathbb{N}$, the *Bernstein basic polynomials* are given as follows

$$b_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0, 1, \dots, n,\ x \in [0,1],$$

the *classical Bernstein operators* $B_n \colon \mathbb{C}([0,1]) \to \mathbb{C}([0,1])$, are defined by

$$(B_n f)(x) = \sum_{i=0}^{n} b_{n,i}(x) f\left(\tfrac{i}{n}\right), \quad x \in [0,1].$$

The following inequality was conjectured as an open problem by I. Raşa in [3],

$$\sum_{i,j=0}^{n} (b_{n,i}(x)b_{n,j}(x) + b_{n,i}(y)b_{n,j}(y) - 2b_{n,i}(x)b_{n,j}(y))f\Big(\frac{i+j}{2n}\Big) \geq 0 \qquad (1)$$

for each convex function $f \in \mathbb{C}([0,1])$ and for all $x, y \in [0,1]$. The proof of inequality (1) was given in [2]. Raşa [4] remarked, that (1) is equivalent to

$$(B_{2n}f)(x) + (B_{2n}f)(y) \geq 2 \sum_{i=0}^{n}\sum_{j=0}^{n} b_{n,i}(x)b_{n,j}(y)\ f\Big(\frac{i+j}{2n}\Big). \qquad (2)$$

In [1] we give some generalizations of inequality (2).

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Ioan Raşa *Functional equations and inequalities for the index of coincidence*

Let $(p_0(x), p_1(x), \ldots)$ be a probability distribution depending on a real parameter x. The associated index of coincidence is $S(x) := \sum_{k=0}^{\infty}(p_k(x))^2$. The Rényi entropy and the Tsallis entropy are defined by $R(x) := -\log S(x)$ and $T(x) := 1 - S(x)$. Starting with the binomial distribution, we establish functional equations and inequalities and use them to investigate convexity properties of the functions $S(x)$, $R(x)$ and $T(x)$. Applications and new open problems are mentioned.

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Debmalya Sain *Norm attainment set of a bounded linear operator between Banach spaces*

It is a topic of current interest in the geometry of Banach spaces to study the norm attainment set of a bounded linear operator between Banach spaces. In this talk, I would like to explore the various facets of this problem, including the case of bounded linear operators between Hilbert spaces and Banach spaces. We would show that it is possible to completely characterize Euclidean spaces among Minkowski spaces, in terms of the operator norm attainment set. We would further explore the norm attainment set of a bounded linear operator between Banach spaces. Using the concept of Birkhoff-James orthogonality and semi-inner-products in Banach spaces, we completely characterize the operator norm attainment set in the setting of Banach spaces. If time permits, we would also like to briefly mention the various areas of application of the norm attainment set of a bounded linear operator, including the study of extreme contractions.

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Ekaterina Shulman *Polynomial-coefficient generalizations of the Levi-Civita and Wilson functional equations*

THEOREM
If continuous functions $f_1, \dots, f_M \colon \mathbb{R} \to \mathbb{C}$ satisfy the functional equation

$$\sum_{i=0}^{M} f_i(x + b_i y) P_i(x, y) = \sum_{j=1}^{n} u_j(x) v_j(y), \quad b_i \neq b_j \text{ for } i \neq j \tag{1}$$

with some polynomials P_i and some continuous functions u_j, v_j, then each f_i is a ratio of an exponential polynomial and a polynomial

$$f_i(x) = \sum_{j=1}^{n} e^{\lambda_j x} r_j(x),$$

where r_j are rational functions.

We discuss also some generalizations of equation (1).

Justyna Sikorska *Various notions of orthogonality and the Cauchy functional equation*

On the example of the famous Cauchy functional equation we show how various notions of orthogonality appear in the theory of functional equations. We give solutions of the Cauchy equation postulated for orthogonal vectors. Applications of this conditional equation both inside and outside mathematics constitutes a significant part of the lecture. Furthermore, we plan to discuss various aspects of stability problem. Last, but not least, some open problems concerning the topic will be presented.

Slavko Simić *Stolarsky means in many variables*

There is a huge amount of papers investigating properties of the so-called Stolarsky (or extended) two-parametric mean value, defined for positive values of $x, y; x \neq y$, as

$$E_{r,s}(x, y) := \left(\frac{r(x^s - y^s)}{s(x^r - y^r)} \right)^{1/(s-r)}, \quad rs(r - s) \neq 0.$$

Those means can be continuously extended on the domain $\{(r,s;x,y):\ \ r,s\in\mathbb{R};\ x,y\in\mathbb{R}_+\}$ by the following

$$E_{r,s}(x,y)=\begin{cases}\left(\dfrac{r(x^s-y^s)}{s(x^r-y^r)}\right)^{1/(s-r)}, & rs(r-s)\neq 0;\\ \exp\left(-\dfrac{1}{s}+\dfrac{x^s\log x-y^s\log y}{x^s-y^s}\right), & r=s\neq 0;\\ \left(\dfrac{x^s-y^s}{s(\log x-\log y)}\right)^{1/s}, & s\neq 0,\ r=0;\\ \sqrt{xy}, & r=s=0;\\ x, & y=x>0,\end{cases}$$

and in this form are introduced by Keneth Stolarsky in [1]. There are several papers attempting to define an extension of the class E to n, $n>2$ variables [1].

In this talk we shall expose two possible explicit formulae of Stolarsky means in n variables which preserve its main properties and coincide for $n=2$.

Definition 1
Let $X_n=(x_1,x_2,...,x_n)\in\mathbb{R}^n_+$. Then,

$$e_{r,s}(X_n)=e_{r,s}(x_1,x_2,...,x_n):=\left(\frac{r^2}{s^2}\frac{x_1^{ns}+x_2^{ns}+...+x_n^{ns}-n(x_1x_2...x_n)^s}{x_1^{nr}+x_2^{nr}+...+x_n^{nr}-n(x_1x_2...x_n)^r}\right)^{\frac{1}{n(s-r)}}$$

for $rs(s-r)\neq 0$.

Definition 2
Let $A_n=(a_1,a_2,...,a_n), X_n=(x_1,x_2,...,x_n), Y_n=(y_1,y_2,...,y_n); A_n,X_n,Y_n\in\mathbb{R}^n_+$. Then,

$$E^n_{r,s}(A_n;X_n,Y_n):=\left(\frac{r^2}{s^2}\frac{a_1(x_1^s-y_1^s)^2+a_2(x_2^s-y_2^s)^2+\cdots+a_n(x_n^s-y_n^s)^2}{a_1(x_1^r-y_1^r)^2+a_2(x_2^r-y_2^r)^2+\cdots+a_n(x_n^r-y_n^r)^2}\right)^{\frac{1}{2(s-r)}}.$$

Both extensions are symmetric and monotone increasing in both parameters r and s with $e_{r,s}(x_1,x_2)=E^1_{r,s}(a_1;x_1,x_2)=E_{r,s}(x_1,x_2)$.

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Peter Stadler *The short ruler on the general affine group*

The restriction to the interval $[0,1]$ of a homomorphism $h\colon (\mathbb{R},+) \to (G,\circ)$ on a Lie group G is a geodesic. The problem is to construct long geodesics. We assume that we have a short ruler, which allows constructing geodesics with length $L > 0$. We can shorten a curve α on G using the short ruler (reduced transformation).

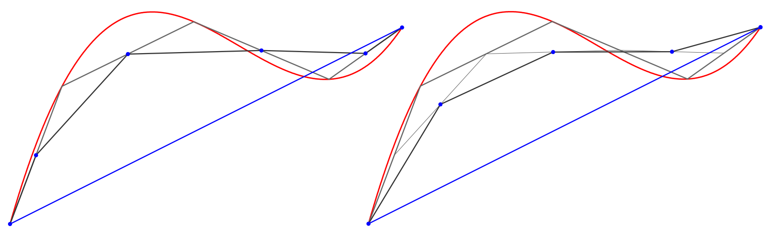

Fig. 1: The reduced transformation.

The reduced process $(R_L^t \alpha)_{t \in \mathbb{N}}$ is the iteration of this transformation. In normed vector spaces, the reduced process converges to the straight line. On the general affine group $\mathrm{Aff}(1,\mathbb{R})$ – which is a Lie group – the reduced process converges to the geodesic linking the starting point of the curve α with its end point.

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Henrik Stetkær *The Small Dimension Lemma revisited*

Dilian Yang [Y] used the Small Dimension Lemma about irreducible, unitary representations of a compact group to solve d'Alembert's and Wilson's functional equations on such a group.

We present a purely algebraic generalization of the Small Dimension Lemma. By help of it we find on any compact group G the solutions $f, g \in C(G)$ of generalizations of d'Alembert's and Wilson's functional equations of the form

$$f(xy) + \mu(y) f(xy^*) = 2f(x)g(y), \quad x, y \in G,$$

where $\mu \in C(G)$ is a given character of G, and $x \mapsto x^*$ is a given, continuous involution of G.

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Tomasz Stypuła *Orthogonality preserving property on small sets*

In this report we consider a problem when the orthogonality preserving property of a linear mapping on a small set, implies its orthogonality preserving property on the whole space. In order to construct such sets, we introduce a concept of independent bases. We present examples and results in finite-dimensional real inner product spaces.

Mariusz Sudzik *Iterative functional equations and attractive fixed points*

Let I be a nontrivial interval and $f, g\colon I \to I$ be given functions. We will consider the functional equation

$$\varphi(x) = \frac{\varphi\big(f(x)\big) + \ \varphi\big(g(x)\big)}{2}$$

under the assumption that f and g have a globally attractive fixed point. Equations of higher order and their inhomogeneous versions will be analysed as well.

László Székelyhidi *Functional equations on infinite joins*
(joint work with Żywilla Fechner)

In this talk we present some results about basic function classes on infinite hypergroup joins. These results may serve as a starting point to study convolution type functional equations on infinite hypergroup joins using spectral synthesis.

Imke Toborg *On the functional equation* $f(x)^{-1} = f^{-1}(x)$ *on groups*

In [1] David J. Schmitz introduced the notion of an inverse ambiguous function. A bijective function from a group into itself is inverse ambiguous if and only if it is a solution of $f(x)^{-1} = f^{-1}(x)$. In this talk we give a precise description when a finite group admits an inverse ambiguous function or an inverse ambiguous automorphism.

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Anita Tomar *Fixed point and applications of Hardy-Rogers type contraction*

In physical world a fixed point signifies a condition wherever a stable state or equilibrium is attained. Presence of fixed point plays a significant role in nonlinear analysis as numerous real-world problems in applied science, economics, chemistry, physics, computer science and engineering can be reformulated as a problem of finding fixed points of nonlinear maps. The aim of this talk is to discuss the existence of fixed point of almost alpha-Hardy-Rogers-$\mathcal{F}-$contraction in a partial metric space. Finally, some interesting examples and the solution of differential equation arising in critically damped harmonic oscillator is also discussed to demonstrate the usability of results.

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Andrzej Wiśnicki *Around the nonlinear Ryll-Nardzewski theorem*

In this talk, we show that if S is a distal semigroup of nonexpansive mappings acting on a weak* compact convex subset Q of a dual Banach space with the Radon-Nikodým property, then there is a common fixed point of S in Q. In particular, it gives a nonlinear counterpart of the Ryll-Nardzewski theorem. As a consequence, we obtain a nonlinear extension of the Bader-Gelander-Monod theorem concerning isometries in L-embedded Banach spaces.

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Alfred Witkowski *Inequalities of Levin-Stečkin and Clausing*

The Levin-Stečkin inequality comes from the Appendix to the Russian translation of "Inequalities" [3].

THEOREM 1 (Levin-Stečkin's inequality)
If a function $p\colon [0,1] \to \mathbb{R}$ satisfies the conditions

- *p is nondecreasing in $[0, 1/2]$,*
- *p is symmetric, i.e. $p(x) = p(1-x)$,*

then for every convex function φ the following inequality holds

$$\int_0^1 p(x)\varphi(x)\mathrm{d}x \le \int_0^1 p(x)\mathrm{d}x \int_0^1 \varphi(x)\mathrm{d}x.$$

In 1980 Clausing [2] proved the following theorem.

Theorem 2 (Clausing's inequality)
Let p be a nonnegative functions on $[0,1]$ satisfying the following conditions:

- *p is nondecreasing on $[0,1/2]$,*
- *p is symmetric.*

Then for every concave, positive function φ the inequality

$$\int_0^1 p(x)\varphi(x)\mathrm{d}x \le \int_0^1 4\min\{x,1-x\}p(x)\mathrm{d}x \int_0^1 \varphi(x)\mathrm{d}x$$

holds.

We provide new, elementary proofs of the above theorems.

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Paweł Wójcik *Semi-smooth points in space $\mathcal{K}(H_1,H_2)$*

The investigations of the smooth points in the operator spaces $\mathcal{K}(H)$ were started in [1]. The aim of this report is to discuss a characterization of semi-smooth points in the compact operator space $\mathcal{K}(H_1,H_2)$, where H_1, H_2 are Hilbert spaces.

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Sebastian Wójcik *On convexity of the Swiss premium principle*

The Swiss premium principle for a risk, represented by a nonnegative bounded random variable X on a given probability space, is defined as a unique real number $H_{u,c}(X)$ satisfying equation

$$u((c-1)H_{u,c}(X)) = E[u(cH_{u,c}(X)-X)] \tag{1}$$

where $c\in[0,1]$ and $u\colon \mathbb{R}\to\mathbb{R}$ is a strictly increasing continuous function with $u(0)=0$ (cf. [1]). In the particular cases $c=0$ and $c=1$ the Swiss premium principle reduces to *the mean-value principle* and *the zero utility principle*, respectively.

In the talk, applying some results in [2, 3, 4], we present a characterization of convexity of the functional $H_{u,c}$ defined implicitly by (1).

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Amr Zakaria *Equality problems related to Cauchy means*
(joint work with Zsolt Páles)

In this talk we establish a new characterization of the equality of two-variable Cauchy means (cf. [4]) to two-variable quasi-arithmetic means (cf. [1]) under natural, and therefore, the weakest possible regularity conditions. As an immediate application, we shed new light on the equality problem of two-variable Cauchy means which was solved by Losonczi [3, Theorem 5] under seven times differentiablity assumptions. The approach is based on the results of the paper [2].

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Thomas Zürcher *Around a Kazimierz Nikodem result - part II*
(joint work with Janusz Morawiec)

This is joint work with Janusz Morawiec, and he delivered part I. In the first part, equations of the form

$$\varphi(x) = \sum_{n=1}^{N} |f_n'(x)|\varphi(f_n(x)) + g(x)$$

were considered. In this talk, we are changing the derivatives f_n' to some other functions g_n, looking for solutions $\varphi \in L^1([0,1])$ of

$$\varphi(x) = \sum_{n=1}^{N} |g_n(x)|\varphi(f_n(x)) + g(x).$$

This is not only a cosmetic change. We need new methods to tackle this kind of equations.

Marcin J. Zygmunt *Additive functions on "ax+b" group*

The aim of the talk is to solve Pexider's equation $f(xy) = g(x)\,h(y)$ for functions $f, g, h\colon G \to G$ acting on a noncommutative group G. The equation will be completely solved in the case of affine group "ax+b".

2. Problems and Remarks

1. Remark

This remark is related to a joint work with Moshe Goldberg and the talk delivered during the 57th Symposium on Functional Equations in Jastarnia, Poland (June 2-9, 2019), based on papers [1, 2].

For a (real or complex) vector space X we use the standard notions of a norm and a seminorm (the latter need not be positive definite). Moreover, a non-zero seminorm which is not a norm is called a *proper seminorm*.

We consider the continuity of seminorms with respect to norm-generated topologies. It can be noticed that a seminorm can be either ubiquitously continuous or ubiquitously discontinuous with respect to any norm-topology. As proved in [1], for a finite-dimensional space X any seminorm S on X is continuous with respect to the unique norm-topology on X (in other words S is continuous with respect to any norm on X). For infinite-dimensional spaces it was proved in [2] that for any non-zero seminorm S there exists a norm with respect to which S is everywhere continuous and there exists a norm with respect to which S is everywhere discontinuous.

Now, one could raise a question whether it can happen that for some norm N on an infinite-dimensional space X all the seminorms defined on X are continuous with respect to the topology generated by N. It turns out, however, that the answer to the above question is negative. Each norm on an infinite-dimensional space admits a proper seminorm which is discontinuous in the topology of the original norm. Moreover, each norm on an infinite-dimensional space admits another norm, discontinuous with respect to the original one.

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Jacek Chmieliński

2. Problem

Let $I \subset \mathbb{R}$ be an interval, $n \in \mathbb{N}$, and $\mathbf{M}\colon I^n \to I^n$ be a mean-type mapping, i.e. $\min(x_1, \dots, x_n) \le \mathbf{M}_k(x_1, \dots, x_n) \le \max(x_1, \dots, x_n)$ for all $(x_1, \dots, x_n) \in I^n$. Moreover assume that each $\mathbf{M}_k$ is symetric. A mean $K\colon I^n \to I$ is called $\mathbf{M}$-invariant if $K \circ \mathbf{M} = K$.

CONJECTURE
Let $n \in \mathbb{N}$ and $\mathbf{M}\colon I^n \to I^n$ be a mean-type mapping such that

$$\max \mathbf{M}(v) - \min \mathbf{M}(v) < \max v - \min v$$

for every nonconstant vector $v \in I^n$. Then there exists at most one continuous $\mathbf{M}$*-invariant mean.*

Recently it was proved that for $n = 2$ this statement is valid.

Paweł Pasteczka

3. Problem

Let B_n be the classical Bernstein operators defined by

$$B_n f(x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) b_{n,k}(x), \quad f \in C[0,1],\ x \in [0,1],$$

where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k = 0, 1, \ldots, n$.

1. Let $(a_k)_{k=0,1,\ldots,n}$ be a convex sequence of nonnegative numbers, i.e. $2a_k \le a_{k-1} + a_{k+1},\ k = 1, \ldots, n-1$. Consider the piecewise linear function $w_n \in C[0,1]$ with $w_n\left(\frac{2k-1}{2n}\right) = 0$, for $k = 1, \ldots, n$, and $w_n\left(\frac{k}{n}\right) = a_k$ for $k = 0, 1, \ldots, n$.

Conjecture 1

$B_{2n} w_n$ is a convex function.

2. Let $r \in [0,1)$ be given. Consider the function

$$F_{n,r}(x) := \sum_{k=0}^{n} b_{n,k}(x) b_{n,k}(x-r), \quad x \in [r,1].$$

Conjecture 2

$F_{n,r}$ is a log*-convex function.*

3. C.A. Micchelli suggested the following property

$$f \in C[0,1] \text{ log-concave } \Rightarrow\ B_n f \text{ log-concave}, \quad n \ge 1.$$

A proof was given by T.N.T. Goodman in 1989. A stronger property is expressed as follows

Conjecture 3

$$f \in C[0,1] \textit{ log-concave } \Rightarrow \sum_{\substack{i+j=h\\ 0\le i\le n-1\\ 0\le j\le n}} \binom{n-1}{i}\binom{n}{j}\Big((n-1-i) f\Big(\frac{j}{n}\Big)\Delta^2_{\frac{1}{n}} f\Big(\frac{i}{n}\Big) - (n-j)\Delta^1_{\frac{1}{n}} f\Big(\frac{i}{n}\Big)\Delta^1_{\frac{1}{n}} f\Big(\frac{j}{n}\Big)\Big) \le 0,$$

for all $n \ge 1$, $h \in \{0, 1, \ldots, 2n-2\}$.

Ioan Raşa

4. Problem

Let G be a commutative topological group. Find the continuous linearly independent solutions $f, g\colon G \to \mathbb{C}$ of the functional equation

$$f(x)[g(x+y) + g(x-y)] = g(x)[f(x+y) + f(x-y)].$$

This functional equation is related to an equation of M. Pompeiu (see *Sur une équation fonctionelle*, C. R. Acad. Sci. Paris **190**, p. 1107, 1930.) The equation can be solved on $G = \mathbb{R}$ if f, g are twice differentiable, by reducing it to a differential equation. It is not known whether all solutions of this equation (under the given conditions) are generalized exponential polynomials.

László Székelyhidi

5. Problem

Let G be a commutative topological group. The subspace V in the space $\mathcal{C}(G)$ of all continuous complex valued functions is called *bi-translation invariant* if for each f in V we have that the function $x \mapsto f(x+y) + f(x-y)$ is in V for each y in V. Clearly, if V is translation invariant and linear, then it is bi-translation invariant, but the converse is not necessarily true. The problem is to describe bi-translation invariant linear spaces of functions which are closed with respect to compact convergence and possibly have some additional properties, like finite dimensionality, etc.

László Székelyhidi

6. Remark (to the Problem of Jose Maria Almira)

The following question was asked in [1]:

Assume that the restriction of a function $f \colon \mathbb{R}^2 \to \mathbb{R}$ to any line $ax + by = 0$ is an exponential polynomial. Is it true that f is an exponential polynomial in two variables?

Actually, J. M. Almira asked a slightly more general question but we presented it in a simpler way (for $n = 2$ instead of an arbitrary n) during the 17th ICFEI.

It turned out that this problem, in much more general settings, was solved by A. L. Ronkin. In particular, he proved the following result.

Corollary 1 ([2])
Let a function f of two real variables be an exponential polynomial when one of the variables is fixed, and assume that $f(x, x+h)$ is an exponential polynomial of x, for each $h \in M$, where M is a set of cardinality continuum. Then $f(x, y)$ is an exponential polynomial in x, y.

The details can be found in [3].

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