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Zakaria Malki, Farida Berhoun and Abdelghani Ouahab

System of boundary random fractional differential equations via Hadamard derivative

Abstract. We study the existence of solutions for random system of fractional differential equations with boundary nonlocal initial conditions. Our approach is based on random fixed point principles of Schaefer and Perov, combined with a vector approach that uses matrices that converge to zero. We prove existence and uniqueness results for these systems. Some examples are presented to illustrate the theory.

1. Introduction

In the previous few decades non-integer order differential and integral equations were given much attention. Because fractional differential equations rise up certainly in various fields and have been applied in physics, chemistry, biology and engineering and has emerged as an important area of investigation in the last few decades. For some fundamental results in the theory of fractional calculus and fractional differential equations see [1, 4, 7, 13, 14, 15].

Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians. We refer the reader to the monographs [5, 12, 25]. The initial value problems for fractional differential equations with random parameters have been studied by Lupulescu and Ntouyas [16]. The basic tool in the study of the problems for random fractional differential equations is to treat it as a fractional differential equation in some appropriate Banach space.

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In [17] authors proved the existence of results for a random fractional equation under a Carathéodory condition. In this paper, we consider the systems of random fractional differential equations with boundary conditions in the following form:

$$\begin{cases} D^{\alpha}(D^{\beta}+\lambda_1)x(t,\omega)=f(t,x(t,\omega),y(t,\omega),\omega) \\ D^{\gamma}(D^{\sigma}+\lambda_2)y(t,\omega)=g(t,x(t,\omega),y(t,\omega),\omega) \\ \sum\limits_{i=1}^{m}\theta_i I^{\mu_i}x(\eta_i,\omega)=\sum\limits_{j=1}^{n}\phi_j I^{\gamma_j}x(\xi_j,\omega) \\ \sum\limits_{k=1}^{p}\varepsilon_k I^{\varsigma_k}x(\psi_k,\omega)=\sum\limits_{l=1}^{q}\nu_l I^{\tau_l}x(\varphi_l,\omega) \\ \sum\limits_{i=1}^{m}\overline{\theta_i} I^{\overline{\mu_i}}y(\overline{\eta_i},\omega)=\sum\limits_{j=1}^{n}\overline{\phi_j} I^{\overline{\gamma_j}}y(\overline{\xi_j},\omega) \\ \sum\limits_{k=1}^{p}\overline{\varepsilon_k} I^{\overline{\varsigma_k}}y(\overline{\psi_k},\omega)=\sum\limits_{l=1}^{q}\overline{\nu_l} I^{\overline{\tau_l}}y(\overline{\varphi_l},\omega) \end{cases} \tag{1}$$

where D^{ρ} denotes the Hadamard Caputo-type fractional derivative of order $\rho, \rho \in \{\alpha,\beta,\gamma,\sigma\}$ with $0<\alpha,\beta,\gamma,\sigma<1$, $1<\alpha+\beta<2$, $1<\gamma+\sigma<2$, λ_1,λ_2 are given constants, I^r is the Hadamard fractional integral of order $r>0$, $r\in\{\mu_i,\gamma_j,\varsigma_k,\tau_l,\overline{\mu_i},\overline{\gamma_j},\overline{\varsigma_k},\overline{\tau_l}\}$ the constants $\eta_i,\xi_j,\psi_k,\varphi_l,\overline{\eta_i},\overline{\xi_j},\overline{\psi_k},\overline{\varphi_l}\in(1,e)$ and $\theta_i,\phi_j,\varepsilon_k,\nu_l,\overline{\theta_i},\overline{\phi_j},\overline{\varepsilon_k},\overline{\nu_l}\in\mathbb{R}$ for all $i=1,2,\dots,m$, $j=1,2,\dots,n$, $k=1,2,\dots,p$, $l=1,2,\dots,q$ and $f,g\colon[1,e]\times\mathbb{R}^m\times\mathbb{R}^m\times\Omega\to\mathbb{R}^m$, $(\Omega,\mathcal{A})$ is a measurable space.

In 2008, Precup [22] proved the role of matrix convergence in the study of semilinear operator systems. Recently, many authors studied the existence of solutions for systems of differential equations and fractional differential equations and inclusions by using vector version of fixed point theorems; see [2, 3, 6, 9, 20, 18, 19] and the references therein.

The paper is organized as follows. In Sections 2 and 3, we set up the appropriate framework on random processes and random fixed point theorems and some fixed point theorems. In Section 4 we give one of our main existence results for solutions and compactness of solutions sets of the problem (1), with the proofs based on recent random fixed point theorems.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $(\widetilde{\Omega},\mathcal{F})$ be a measurable space; that is, a set $\widetilde{\Omega}$ with a σ-algebra of subsets of $\widetilde{\Omega}$. A probability measure $\mathbb{P}$ is a measure on with $\mathbb{P}(\Omega)=1$. Then $(\Omega,\mathcal{F},\mathbb{P})$ is called a probability space. In the following, assume that $(\widetilde{\Omega},\mathcal{F},\mathbb{P})$ is a complete probability space. Let X be a metric space, $B(X)$ will be the σ-algebra of all Borel subsets of X. A measurable function $x\colon\widetilde{\Omega}\to X$ is called a random element in X. A random element in X is called a random variable.

Let X,Y be two locally compact, metric spaces and $f\colon\widetilde{\Omega}\times X\to Y$. By $C(X,Y)$ we denote the space of continuous functions from X into Y endowed with the compact-open topology.

Definition 2.1 ([10])
For at least n-times differentiable function $y\colon [1,\infty) \to \mathbb{R}$, the Caputo-type Hadamard derivative of fractional order α is defined as

$$D^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{n-\alpha-1}\delta^n y(s)\frac{s}{ds}, \qquad n-1<\alpha<n,\ n=[\alpha]+1,$$

where $\delta = t\frac{d}{dt}$, $[\alpha]$ denotes the integer part of the real number α and $\log(\cdot) = \log_e(\cdot)$

Definition 2.2 ([11])
The Hadamard fractional integral of order α for a function y is defined as

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{y(s)}{s}ds, \qquad \alpha>0,$$

provided the integral exists.

Lemma 2.3 ([10])
Let $u \in AC^n_\delta[a,b]$ or $C^n_\delta[a,b]$ and $\alpha \in \mathbb{C}$, where $X^n_\delta[a,b] = \{f\colon [a,b] \to \mathbb{C} : \delta^{(n-1)}f(t) \in X[a,b]\}$. Then, we have

$$I^{\alpha}(D^{\alpha})u(t) = u(t) - \sum_{k=0}^{n-1} c_k(\log t)^k,$$

where $c_i \in \mathbb{R}$, $i = 1,2,\dots,n-1$ $(n=[\alpha]+1)$.

For convenience, we set constants

$$\begin{aligned}
\Omega_1 &= \sum_{i=1}^{m}\theta_i\frac{(\log\eta_i)^{\mu_i}}{\Gamma(\mu_i+1)} - \sum_{j=1}^{n}\phi_j\frac{(\log\xi_j)^{\gamma_j}}{\Gamma(\gamma_j+1)},\\
\Omega_2 &= \sum_{i=1}^{m}\theta_i\frac{(\log\eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)} - \sum_{j=1}^{n}\phi_j\frac{(\log\xi_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)},\\
\Omega_3 &= \sum_{k=1}^{p}\varepsilon_k\frac{(\log\psi_k)^{\varsigma_k}}{\Gamma(\varsigma_k+1)} - \sum_{l=1}^{q}\nu_l\frac{(\log\varphi_l)^{\tau_l}}{\Gamma(\tau_l+1)},\\
\Omega_4 &= \sum_{k=1}^{p}\varepsilon_k\frac{(\log\psi_k)^{\beta+\varsigma_k}}{\Gamma(\beta+\varsigma_k+1)} - \sum_{l=1}^{q}\nu_l\frac{(\log\varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)}
\end{aligned}$$

and

$$\Omega = \Omega_1\Omega_4 - \Omega_2\Omega_3.$$

Similarly, we set

$$\overline{\Omega_1} = \sum_{i=1}^{m}\overline{\theta_i}\frac{(\log\overline{\eta_i})^{\overline{\mu_i}}}{\Gamma(\overline{\mu_i}+1)} - \sum_{j=1}^{n}\overline{\phi_j}\frac{(\log\overline{\xi_j})^{\overline{\gamma_j}}}{\Gamma(\overline{\gamma_j}+1)},$$

$$\overline{\Omega_2} = \sum_{i=1}^{m} \overline{\theta_i} \frac{(\log \overline{\eta_i})^{\sigma+\overline{\mu_i}}}{\Gamma(\sigma+\overline{\mu_i}+1)} - \sum_{j=1}^{n} \overline{\phi_j} \frac{(\log \overline{\xi_j})^{\sigma+\overline{\gamma_j}}}{\Gamma(\sigma+\overline{\gamma_j}+1)},$$

$$\overline{\Omega_3} = \sum_{k=1}^{p} \overline{\varepsilon_k} \frac{(\log \overline{\psi_k})^{\overline{\varsigma_k}}}{\Gamma(\overline{\varsigma_k}+1)} - \sum_{l=1}^{q} \overline{\nu_l} \frac{(\log \overline{\varphi_l})^{\overline{\tau_l}}}{\Gamma(\overline{\tau_l}+1)},$$

$$\overline{\Omega_4} = \sum_{k=1}^{p} \overline{\varepsilon_k} \frac{(\log \overline{\psi_k})^{\sigma+\overline{\varsigma_k}}}{\Gamma(\sigma+\overline{\varsigma_k}+1)} - \sum_{l=1}^{q} \overline{\nu_l} \frac{(\log \overline{\varphi_l})^{\sigma+\overline{\tau_l}}}{\Gamma(\sigma+\overline{\tau_l}+1)}$$

and

$$\overline{\Omega} = \overline{\Omega_1 \Omega_4} - \overline{\Omega_2 \Omega_3}.$$

Lemma 2.4 ([21])
f is a Carathéodory function if and only if $\omega \to r(\omega)(\cdot) = f(\omega, \cdot)$ is a measurable function from $\widetilde{\Omega}$ to $C(X, Y)$.

Proposition 2.1
If $f\colon [1, e] \times \widetilde{\Omega} \to \mathbb{R}^m$ is a Carathéodory function, then the function $(t, \omega) \mapsto I^\alpha f(t, \omega)$ is also a Carathéodory function.

Proof. Clear that $I^\alpha\colon C([1, e], \mathbb{R}^m) \to \mathbb{R}^m$ is a continuous operator, let $L\colon \Omega \to C([1, e], \mathbb{R}^m)$ be defined by $L(\omega)(\cdot) = f(\cdot, \omega)$. From Lemma 2.4, $L(\cdot)$ is measurable. Then the operator $\omega \to (I^\alpha \circ L)(\omega)(\cdot)$ is measurable. The function $t \mapsto I^\alpha f(t, \omega)$ is a continuous function. Hence $(t, \omega) \mapsto I^\alpha f(t, \omega)$ is a Carathéodory function.

Lemma 2.5 ([24])
Let $\widetilde{\Omega} \neq 0$, $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, λ_1 is a given constant, $\mu_i, \gamma_j, \varsigma_k, \tau_l > 0$, $\eta_i, \xi_j, \psi_k, \varphi_l \in (1, e)$ and $\theta_i, \phi_j, \varepsilon_k, \nu_l \in \mathbb{R}$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, p$ and $l = 1, 2, \ldots, q$. Then the problem

$$\begin{cases} D^\alpha (D^\beta + \lambda_1) x(t, \omega) = f(t, x(t, \omega), y(t, \omega), \omega) \\ \sum\limits_{i=1}^{m} \theta_i I^{\mu_i} x(\eta_i, \omega) = \sum\limits_{j=1}^{n} \phi_j I^{\gamma_j} x(\xi_j, \omega) \\ \sum\limits_{k=1}^{p} \varepsilon_k I^{\varsigma_k} x(\psi_k, \omega) = \sum\limits_{l=1}^{q} \nu_l I^{\tau_l} x(\varphi_l, \omega) \end{cases}$$

has a unique solution given by

$$\begin{aligned} x(t) = \frac{1}{\Omega} \Bigg[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_3 \right) & \\ \times \Bigg(\sum_{j=1}^{n} \phi_j [I^{\alpha+\beta+\gamma_j} f(\xi_j, x(\xi_j, \omega), y(\xi_j, \omega), \omega) - \lambda_1 I^{\beta+\gamma_j} x(\xi_j, \omega)] & \\ - \sum_{i=1}^{m} \theta_i [I^{\alpha+\beta+\mu_i} f(\eta_i, x(\eta_i, \omega), y(\eta_i, \omega), \omega) - \lambda_1 I^{\beta+\mu_i} x(\eta_i, \omega)] \Bigg) & \end{aligned}$$

$$+\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right)$$
$$\times\Bigg(\sum_{l=1}^{q}\upsilon_l[I^{\alpha+\beta+\tau_l}f(\varphi_l,x(\varphi_l,\omega),y(\varphi_l,\omega),\omega)-\lambda_1 I^{\beta+\tau_l}x(\varphi_l,\omega)]$$
$$-\sum_{k=1}^{p}\varepsilon_k[I^{\alpha+\beta+\varsigma_k}f(\psi_k,x(\psi_k,\omega),y(\psi_k,\omega),\omega)-\lambda_1 I^{\beta+\varsigma_k}x(\psi_k,\omega)]\Bigg)\Bigg]$$
$$+I^{\alpha+\beta}f(t,x(t,\omega),y(t,\omega),\omega)-\lambda_1 I^{\beta}x(t,\omega).$$

Similarly, let $\overline{\Omega}\neq 0, 0<\gamma,\sigma\leq 1$, $1<\gamma+\sigma\leq 2$, λ_2 *is a given constant,* $\overline{\mu_i},\overline{\gamma_j},\overline{\varsigma_k},\overline{\tau_l}>0$, $\overline{\eta_i},\overline{\xi_j},\overline{\psi_k},\overline{\varphi_l}\in(1,e)$ *and* $\overline{\theta_i},\overline{\phi_j},\overline{\varepsilon_k},\overline{\nu_l}\in\mathbb{R}$ *for* $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, $k=1,2,\ldots,p$ *and* $l=1,2,\ldots,q$. *Then the problem*

$$\begin{cases}D^{\gamma}(D^{\sigma}+\lambda_2)y(t,\omega)=g(t,x(t,\omega),y(t,\omega),\omega)\\ \sum\limits_{i=1}^{m}\overline{\theta_i}I^{\overline{\mu_i}}y(\overline{\eta_i},\omega)=\sum\limits_{j=1}^{n}\overline{\phi_j}I^{\overline{\gamma_j}}y(\overline{\xi_j},\omega)\\ \sum\limits_{k=1}^{p}\overline{\varepsilon_k}I^{\overline{\varsigma_k}}y(\overline{\psi_k},\omega)=\sum\limits_{l=1}^{q}\overline{\nu_l}I^{\overline{\tau_l}}y(\overline{\varphi_l},\omega)\end{cases}$$

has a unique solution given by

$$y(t)=\frac{1}{\overline{\Omega}}\Bigg[\left(\overline{\Omega_4}-\frac{(\log t)^{\sigma}}{\Gamma(\sigma+1)}\overline{\Omega_3}\right)$$
$$\times\Bigg(\sum_{j=1}^{n}\overline{\phi_j}[I^{\gamma+\sigma+\overline{\gamma_j}}g(\overline{\xi_j},x(\overline{\xi_j},\omega),y(\overline{\xi_j},\omega),\omega)-\lambda_2 I^{\sigma+\overline{\gamma_j}}y(\overline{\xi_j},\omega)]$$
$$-\sum_{i=1}^{m}\overline{\theta_i}[I^{\gamma+\sigma+\overline{\mu_i}}g(\overline{\eta_i},x(\overline{\eta_i},\omega),y(\overline{\eta_i},\omega),\omega)-\lambda_2 I^{\sigma+\overline{\mu_i}}y(\overline{\eta_i},\omega)]\Bigg)$$
$$+\left(\frac{(\log t)^{\sigma}}{\Gamma(\sigma+1)}\overline{\Omega_1}-\overline{\Omega_2}\right)$$
$$\times\Bigg(\sum_{l=1}^{q}\overline{\upsilon_l}[I^{\gamma+\sigma+\overline{\tau_l}}g(\overline{\varphi_l},x(\overline{\varphi_l},\omega),y(\overline{\varphi_l},\omega),\omega)-\lambda_2 I^{\sigma+\overline{\tau_l}}y(\overline{\varphi_l},\omega)]$$
$$-\sum_{k=1}^{p}\overline{\varepsilon_k}[I^{\gamma+\sigma+\overline{\varsigma_k}}g(\overline{\psi_k},x(\overline{\psi_k},\omega),y(\overline{\psi_k},\omega),\omega)-\lambda_2 I^{\sigma+\overline{\varsigma_k}}y(\overline{\psi_k},\omega)]\Bigg)\Bigg]$$
$$+I^{\gamma+\sigma}g(t,x(t,\omega),y(t,\omega),\omega)-\lambda_2 I^{\sigma}y(t,\omega)$$

3. Fixed point theorems

Definition 3.1
A random operator $T\colon \widetilde{\Omega}\times X\to X$ is said to be continuous at $x_0\in X$ if

$$\lim_{n\to\infty}\|x_n-x_0\|=0 \ \text{ implies } \ \lim_{n\to\infty}\|T(\omega,x_n)-T(\omega,x)\|=0 \ \text{ a.s.}$$

Definition 3.2
A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, all the eigenvalues of M are in the open unit disc.

Theorem 3.3 ([26], pages 12, 88)
Let $M\in\mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:

(i) *M is convergent towards zero;*

(ii) *$M^k\to 0$ as $k\to\infty$;*

(iii) *The matrix $(I-M)$ is nonsingular and*

$$(I-M)^{-1}=I+M+M^2+\ldots+M^k+\ldots;$$

(iv) *The matrix $(I-M)$ is nonsingular and $(I-M)^{-1}$ has nonnegative elements.*

Theorem 3.4 ([23])
Let $(\widetilde{\Omega},\mathcal{F},\mu)$ be a probability space, X be a real separable generalized Banach space and $T\colon\widetilde{\Omega}\times X\to X$ be a continuous random operator. Let $M(\omega)\in\mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matric such that $M(\omega)$ converge to 0 a.s. and

$$d(T(\omega,x_1),T(\omega,x_2))\le M(\omega)d(x_1,x_2) \qquad \text{for each } x_1,x_2\in X,\omega\in\widetilde{\Omega}.$$

Then there exists a random variable and $x:\widetilde{\Omega}\to X$ which is the unique random fixed point of T.

Theorem 3.5 ([23])
Let $(\widetilde{\Omega},\mathcal{F})$ be a measurable space, X be a real separable generalized Banach space and $T\colon\widetilde{\Omega}\times X\to X$ be a continuous random operator and let $M(\omega)\in M_{n\times n}(\mathbb{R}_+)$ be a random variable matrix such that for every $\omega\in\widetilde{\Omega}$ the matrix $M(\omega)$ converge to 0 a.s.,

$$d(T(\omega,x_1),T(\omega,x_2))\le M(\omega)d(x_1,x_2) \qquad \text{for each } x_1,x_2\in X,\omega\in\widetilde{\Omega}.$$

Then there exists a random variable and $x\colon\widetilde{\Omega}\to X$ which is the unique random fixed point of T.

Theorem 3.6 ([8, 23])
Let X be a real separable generalized Banach space and $T\colon\widetilde{\Omega}\times X\to X$ be a completely continuous random operator. Then, either of the following holds

(i) *The random equation $T(\omega,x)=x$ has a random solution, i.e. there is a measurable function $x\colon\widetilde{\Omega}\to X$ such that $T(\omega,x(\omega))=x(\omega)$ for all $\omega\in\widetilde{\Omega}$,*

(ii) *The set $M=\{x:\widetilde{\Omega}\to X$ is measurable $\lambda(\omega)T(\omega,x)=x\}$ is unbounded for some measurable $\lambda\colon\widetilde{\Omega}\to X$ with $0<\lambda(\omega)<1$ on $\widetilde{\Omega}$.*

4. Existence and Uniqueness

Let us set the constants

$$\begin{aligned}\Lambda_1(u) = \frac{1}{|\Omega|}\Bigg[&\Big(|\Omega_4| + \frac{|\Omega_3|}{\Gamma(\beta+1)}\Big)\\ &\times\Bigg(\sum_{j=1}^{n}|\phi_j|\frac{(\log\xi_j)^{u+\beta+\gamma_j}}{\Gamma(u+\beta+\gamma_j+1)}+\sum_{i=1}^{m}|\theta_i|\frac{(\log\eta_i)^{u+\beta+\mu_i}}{\Gamma(u+\beta+\mu_i+1)}\Bigg)\\ &+\Big(\frac{|\Omega_1|}{\Gamma(\beta+1)}+|\Omega_2|\Big)\\ &\times\Bigg(\sum_{l=1}^{q}|\upsilon_l|\frac{(\log\varphi_l)^{u+\beta+\tau_l}}{\Gamma(u+\beta+\tau_l+1)}+\sum_{k=1}^{p}|\varepsilon_k|\frac{(\log\psi_k)^{u+\beta+\varsigma_k}}{\Gamma(u+\beta+\varsigma_k+1)}\Bigg)\Bigg]\\ &+\frac{1}{\Gamma(u+\beta+1)}\end{aligned}$$

and

$$\begin{aligned}\Lambda_2(u) = \frac{1}{|\overline{\Omega}|}\Bigg[&\Big(|\overline{\Omega_4}| + \frac{|\overline{\Omega_3}|}{\Gamma(\sigma+1)}\Big)\\ &\times\Bigg(\sum_{j=1}^{n}|\overline{\phi_j}|\frac{(\log\overline{\xi_j})^{u+\sigma+\overline{\gamma_j}}}{\Gamma(u+\sigma+\overline{\gamma_j}+1)}+\sum_{i=1}^{m}|\overline{\theta_i}|\frac{(\log\overline{\eta_i})^{u+\sigma+\overline{\mu_i}}}{\Gamma(u+\sigma+\overline{\mu_i}+1)}\Bigg)\\ &+\Big(\frac{|\overline{\Omega_1}|}{\Gamma(\sigma+1)}+|\overline{\Omega_2}|\Big)\\ &\times\Bigg(\sum_{l=1}^{q}|\overline{\upsilon}_l|\frac{(\log\overline{\varphi_l})^{u+\sigma+\overline{\tau_l}}}{\Gamma(u+\sigma+\overline{\tau_l}+1)}+\sum_{k=1}^{p}|\overline{\varepsilon_k}|\frac{(\log\overline{\psi_k})^{u+\sigma+\overline{\varsigma_k}}}{\Gamma(u+\sigma+\overline{\varsigma_k}+1)}\Bigg)\Bigg]\\ &+\frac{1}{\Gamma(u+\sigma+1)}.\end{aligned}$$

Our main first result is the existence and uniqueness of random solution of the problem (1).

Theorem 4.1
Let $f,g\colon [1,e]\times\mathbb{R}^m\times\mathbb{R}^m\times\widetilde{\Omega}\to\mathbb{R}^m$ are two Carathéodory functions. Assume that the following condition holds:

(H) *There exist $p_1,p_2,p_3,p_4:\widetilde{\Omega}\to\mathbb{R}_+$ are random variable such that*

$$\forall x,y,\tilde{x},\tilde{y}\in\mathbb{R}^m\quad \|f(t,x,y,\omega)-f(t,\tilde{x},\tilde{y},\omega)\|\le p_1(\omega)\|x-\tilde{x}\|+p_2(\omega)\|y-\tilde{y}\|$$

and

$$\forall x,y,\tilde{x},\tilde{y}\in\mathbb{R}^m\quad \|g(t,x,y,\omega)-g(t,\tilde{x},\tilde{y},\omega)\|\le p_3(\omega)\|x-\tilde{x}\|+p_4(\omega)\|y-\tilde{y}\|.$$

If for every $\omega \in \widetilde{\Omega}, \widetilde{M}(\omega)$ *converge to* 0, *where*

$$\widetilde{M}(\omega) = \begin{pmatrix} \Lambda_1(\alpha)p_1(\omega) + |\lambda_1|\Lambda_1(0) & \Lambda_1(\alpha)p_2(\omega) \\ \Lambda_2(\gamma)p_3(\omega) & \Lambda_2(\gamma)p_4(\omega) + |\lambda_2|\Lambda_2(0) \end{pmatrix},$$

then problem (1) *has unique random solution.*

Proof. Consider the operator $N\colon C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)\times\Omega \to C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$ given by

$$(x(.,\omega), y(.,\omega), \omega) \mapsto (N_1(t, x(t,\omega), y(t,\omega), \omega), N_2(t, x(t,\omega), y(t,\omega), \omega)),$$

where

$$\begin{aligned} N_1&(x(t,\omega), y(t,\omega), \omega) \\ &= \frac{1}{\Omega}\Bigg[\Big(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\Big) \\ &\quad\times\Bigg(\sum_{j=1}^{n}\phi_j[I^{\alpha+\beta+\gamma_j}f(\xi_j, x(\xi_j,\omega), y(\xi_j,\omega),\omega) - \lambda_1 I^{\beta+\gamma_j}x(\xi_j,\omega)] \\ &\quad-\sum_{i=1}^{m}\theta_i[I^{\alpha+\beta+\mu_i}f(\eta_i, x(\eta_i,\omega), y(\eta_i,\omega),\omega) - \lambda_1 I^{\beta+\mu_i}x(\eta_i,\omega)]\Bigg) \\ &\quad+\Big(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\Big) \\ &\quad\times\Bigg(\sum_{l=1}^{q}\upsilon_l[I^{\alpha+\beta+\tau_l}f(\varphi_l, x(\varphi_l,\omega), y(\varphi_l,\omega),\omega) - \lambda_1 I^{\beta+\tau_l}x(\varphi_l,\omega)] \\ &\quad-\sum_{k=1}^{p}\varepsilon_k[I^{\alpha+\beta+\varsigma_k}f(\psi_k, x(\psi_k,\omega), y(\psi_k,\omega),\omega) - \lambda_1 I^{\beta+\varsigma_k}x(\psi_k,\omega)]\Bigg)\Bigg] \\ &\quad+ I^{\alpha+\beta}f(t, x(t,\omega), y(t,\omega),\omega) - \lambda_1 I^\beta x(t,\omega) \end{aligned}$$

and

$$\begin{aligned} N_2&(x(t,\omega), y(t,\omega), \omega) \\ &= \frac{1}{\overline{\Omega}}\Bigg[\Big(\overline{\Omega_4} - \frac{(\log t)^\sigma}{\Gamma(\sigma+1)}\overline{\Omega_3}\Big) \\ &\quad\times\Bigg(\sum_{j=1}^{n}\overline{\phi_j}[I^{\gamma+\sigma+\overline{\gamma_j}}g(\overline{\xi_j}, x(\overline{\xi_j},\omega), y(\overline{\xi_j},\omega),\omega) - \lambda_2 I^{\sigma+\overline{\gamma_j}}y(\overline{\xi_j},\omega)] \\ &\quad-\sum_{i=1}^{m}\overline{\theta_i}[I^{\gamma+\sigma+\overline{\mu_i}}g(\overline{\eta_i}, x(\overline{\eta_i},\omega), y(\overline{\eta_i},\omega),\omega) - \lambda_2 I^{\sigma+\overline{\mu_i}}y(\overline{\eta_i},\omega)]\Bigg) \end{aligned}$$

$$+\Big(\frac{(\log t)^{\sigma}}{\Gamma(\sigma+1)}\overline{\Omega_1}-\overline{\Omega_2}\Big)$$

$$\times\Big(\sum_{l=1}^{q}\overline{\upsilon_l}[I^{\gamma+\sigma+\overline{\tau_l}}g(\overline{\varphi_l},x(\overline{\varphi_l},\omega),y(\overline{\varphi_l},\omega),\omega)-\lambda_2 I^{\sigma+\overline{\tau_l}}y(\overline{\varphi_l},\omega)]$$

$$-\sum_{k=1}^{p}\overline{\varepsilon_k}[I^{\gamma+\sigma+\overline{\varsigma_k}}g(\overline{\psi_k},x(\overline{\psi_k},\omega),y(\overline{\psi_k},\omega),\omega)-\lambda_2 I^{\sigma+\overline{\varsigma_k}}y(\overline{\psi_k},\omega)]\Big)\Big]$$

$$+I^{\gamma+\sigma}g(t,x(t,\omega),y(t,\omega),\omega)-\lambda_2 I^{\sigma}y(t,\omega).$$

First we show that N is a random operator on $C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$. Since f and g are Carathéodory functions, then $\omega\mapsto f(t,x,y,\omega)$ and $\omega\mapsto g(t,x,y,\omega)$ are measurable maps in view of Proposition 2.1 we concluded that, the maps $\omega\mapsto N_1(x(t,\omega),y(t,\omega),\omega)$, $\omega\mapsto N_2(x(t,\omega),y(t,\omega),\omega)$ are measurable. As a result, N is a random operator on $C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)\times\Omega$ into $C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$.

We show that N satisfies all the conditions of Theorem 3.4 on $C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$. Let $(x,y),(\tilde{x},\tilde{y})\in C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$, then

$$\|N_1(x(t,\omega),y(t,\omega),\omega)-N_1(\tilde{x}(t,\omega),\tilde{y}(t,\omega),\omega)\|$$

$$=\Big\|\frac{1}{\Omega}\Big[\Big(\Omega_4-\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\Big)\Big(\sum_{j=1}^{n}\phi_j[I^{\alpha+\beta+\gamma_j}f(\xi_j,x(\xi_j,\omega),y(\xi_j,\omega))$$

$$-I^{\alpha+\beta+\gamma_j}f(\xi_j,\tilde{x}(\xi_j,\omega),\tilde{y}(\xi_j,\omega))-\lambda_1[I^{\beta+\gamma_j}x(\xi_j,\omega)-I^{\beta+\gamma_j}\tilde{x}(\xi_j,\omega)]]$$

$$-\sum_{i=1}^{m}\theta_i[I^{\alpha+\beta+\mu_i}f(\eta_i,x(\eta_i,\omega),y(\eta_i,\omega))-I^{\alpha+\beta+\mu_i}f(\eta_i,\tilde{x}(\eta_i,\omega),\tilde{y}(\eta_i,\omega))$$

$$-\lambda_1[I^{\beta+\mu_i}x(\eta_i,\omega)-I^{\beta+\mu_i}\tilde{x}(\eta_i,\omega)]]\Big)$$

$$+\Big(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1-\Omega_2\Big)\Big(\sum_{l=1}^{q}\upsilon_l[I^{\alpha+\beta+\tau_l}f(\varphi_l,x(\varphi_l,\omega),y(\varphi_l,\omega))$$

$$-I^{\alpha+\beta+\tau_l}f(\varphi_l,\tilde{x}(\varphi_l,\omega),\tilde{y}(\varphi_l,\omega))-\lambda_1[I^{\beta+\tau_l}x(\varphi_l,\omega)-I^{\beta+\tau_l}\tilde{x}(\varphi_l,\omega)]]$$

$$-\sum_{k=1}^{p}\varepsilon_k[I^{\alpha+\beta+\varsigma_k}f(\psi_k,x(\psi_k,\omega),y(\psi_k,\omega))$$

$$-I^{\alpha+\beta+\varsigma_k}f(\psi_k,\tilde{x}(\psi_k,\omega),\tilde{y}(\psi_k,\omega))$$

$$-\lambda_1[I^{\beta+\varsigma_k}x(\psi_k,\omega)-I^{\beta+\varsigma_k}\tilde{x}(\psi_k,\omega)]]\Big)\Big]$$

$$+I^{\alpha+\beta}f(t,x(t,\omega),y(t,\omega))-I^{\alpha+\beta}f(t,\tilde{x}(t,\omega),\tilde{y}(t,\omega))$$

$$
\begin{aligned}
&- \lambda_1[I^{\beta}x(t,\omega) - I^{\beta}\tilde{x}(t,\omega)]\Big\| \\
&\leq \frac{1}{\Omega}\bigg[\Big|\Omega_4 - \frac{(\log t)^{\beta}}{\mathit{\Gamma}(\beta+1)}\Omega_3\Big|\bigg(\sum_{j=1}^{n}\phi_j[I^{\alpha+\beta+\gamma_j}\|f(\xi_j,x(\xi_j,\omega),y(\xi_j,\omega)) \\
&\quad - f(\xi_j,\tilde{x}(\xi_j,\omega),\tilde{y}(\xi_j,\omega))\| + \lambda_1 I^{\beta+\gamma_j}\|x(\xi_j,\omega)-\tilde{x}(\xi_j,\omega)\|] \\
&\quad + \sum_{i=1}^{m}\theta_i[I^{\alpha+\beta+\mu_i}\|f(\eta_i,x(\eta_i,\omega),y(\eta_i,\omega)) - f(\eta_i,\tilde{x}(\eta_i,\omega),\tilde{y}(\eta_i,\omega))\| \\
&\quad + \lambda_1 I^{\beta+\mu_i}\|x(\eta_i,\omega)-\tilde{x}(\eta_i,\omega)\|]\bigg) \\
&\quad + \Big|\frac{(\log t)^{\beta}}{\mathit{\Gamma}(\beta+1)}\Omega_1 - \Omega_2\Big|\bigg(\sum_{l=1}^{q}\upsilon_l[I^{\alpha+\beta+\tau_l}\|f(\varphi_l,x(\varphi_l,\omega),y(\varphi_l,\omega)) \\
&\quad - f(\varphi_l,\tilde{x}(\varphi_l,\omega),\tilde{y}(\varphi_l,\omega))\| + \lambda_1 I^{\beta+\tau_l}\|x(\varphi_l,\omega)-\tilde{x}(\varphi_l,\omega)\|] \\
&\quad + \sum_{k=1}^{p}\varepsilon_k[I^{\alpha+\beta+\varsigma_k}\|f(\psi_k,x(\psi_k,\omega),y(\psi_k,\omega)) - f(\psi_k,\tilde{x}(\psi_k,\omega),\tilde{y}(\psi_k,\omega))\| \\
&\quad + \lambda_1 I^{\beta+\varsigma_k}\|x(\psi_k,\omega)-\tilde{x}(\psi_k,\omega)\|]\bigg)\bigg] \\
&\quad + I^{\alpha+\beta}\|f(t,x(t,\omega),y(t,\omega)) - f(t,\tilde{x}(t,\omega),\tilde{y}(t,\omega))\| \\
&\quad + \lambda_1 I^{\beta}\|x(t,\omega)-\tilde{x}(t,\omega)\| \\
&\leq p_1(\omega)\bigg\{\frac{1}{|\Omega|}\bigg[\Big(|\Omega_4| + \frac{|\Omega_3|}{\mathit{\Gamma}(\beta+1)}\Big) \\
&\quad \times\bigg(\sum_{j=1}^{n}|\phi_j|\frac{(\log\xi_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)} + \sum_{i=1}^{m}|\theta_i|\frac{(\log\eta)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)}\bigg) \\
&\quad + \Big(\frac{(\log t)^{\beta}}{\mathit{\Gamma}(\beta+1)}|\Omega_1| + |\Omega_2|\Big) \\
&\quad \times\bigg(\sum_{l=1}^{q}|\upsilon_l|\frac{(\log\varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)} + \sum_{k=1}^{p}|\varepsilon_k|\frac{(\log\psi_k)^{\alpha+\beta+\varsigma_k}}{\Gamma(\alpha+\beta+\varsigma_k+1)}\bigg)\bigg] \\
&\quad + \frac{1}{\Gamma(\alpha+\beta+1)}\bigg\}\|x(.,\omega)-\tilde{x}(.,\omega)\| \\
&\quad + |\lambda_1|\bigg\{\frac{1}{|\Omega|}\bigg[\Big(|\Omega_4| + \frac{|\Omega_3|}{\mathit{\Gamma}(\beta+1)}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\times \Bigg(\sum_{j=1}^{n} |\phi_j| \frac{(\log \xi_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)} + \sum_{i=1}^{m} |\theta_i| \frac{(\log \eta)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)} \Bigg) \\
&+ \Big(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} |\Omega_1| + |\Omega_2| \Big) \\
&\times \Bigg(\sum_{l=1}^{q} |\upsilon_l| \frac{(\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)} + \sum_{k=1}^{p} |\varepsilon_k| \frac{(\log \psi_k)^{\beta+\varsigma_k}}{\Gamma(\beta+\varsigma_k+1)} \Bigg) \Bigg] \\
&+ \frac{1}{\Gamma(\beta+1)} \Bigg\} \|x(\cdot,\omega) - \tilde{x}(\cdot,\omega)\| \\
&+ p_2(\omega) \Bigg\{ \frac{1}{|\Omega|} \Bigg[\Big(|\Omega_4| + \frac{(\log t)^{\beta}}{\Gamma(\beta+1)} |\Omega_3| \Big) \\
&\times \Bigg(\sum_{j=1}^{n} |\phi_j| \frac{(\log \xi_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)} \sum_{i=1}^{m} |\theta_i| \frac{(\log \eta)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)} \Bigg) \\
&+ \Big(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} |\Omega_1| + |\Omega_2| \Big) \\
&\times \Bigg(\sum_{l=1}^{q} |\upsilon_l| \frac{(\log \varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)} + \sum_{k=1}^{p} |\varepsilon_k| \frac{(\log \psi_k)^{\alpha+\beta+\varsigma_k}}{\Gamma(\alpha+\beta+\varsigma_k+1)} \Bigg) \Bigg] \\
&+ \frac{1}{\Gamma(\alpha+\beta+1)} \Bigg\} \|y(\cdot,\omega) - \tilde{y}(\cdot,\omega)\| \\
&+ |\lambda_1| \Bigg\{ \frac{1}{|\Omega|} \Bigg[\Big(|\Omega_4| + \frac{|\Omega_3|}{\Gamma(\beta+1)} \Big) \\
&\times \Bigg(\sum_{j=1}^{n} |\phi_j| \frac{(\log \xi_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)} + \sum_{i=1}^{m} |\theta_i| \frac{(\log \eta)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)} \Bigg) \\
&+ \Big(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} |\Omega_1| + |\Omega_2| \Big) \\
&\times \Bigg(\sum_{l=1}^{q} |\upsilon_l| \frac{(\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)} + \sum_{k=1}^{p} |\varepsilon_k| \frac{(\log \psi_k)^{\beta+\varsigma_k}}{\Gamma(\beta+\varsigma_k+1)} \Bigg) \Bigg] \\
&+ \frac{1}{\Gamma(\beta+1)} \Bigg\} \|y(\cdot,\omega) - \tilde{y}(\cdot,\omega)\| \\
\le\, & (p_1(\omega)\Lambda_1(\alpha) + |\lambda_1|\Lambda_1(0)) \|x(\cdot,\omega) - \tilde{x}(\cdot,\omega)\|_\infty \\
&+ p_2(\omega)\Lambda(\alpha) \|y(\cdot,\omega) - \tilde{y}(\cdot,\omega)\|_\infty,
\end{aligned}
$$

then

$$\|N_1(t,x,y,\omega)-N_1(t,\tilde{x},\tilde{y},\omega)\|_\infty \leq (p_1(\omega)\Lambda_1(\alpha)+|\lambda_1|\Lambda_1(0))\|x-\tilde{x}\|_\infty+p_2(\omega)\Lambda_1(\alpha)\|y-\tilde{y}\|_\infty.$$

Similarly, we obtain

$$\|N_2(t,x,y,\omega)-N_2(t,\tilde{x},\tilde{y},\omega)\|_\infty \leq p_3(\omega)\Lambda_2(\gamma)\|x-\tilde{x}\|_\infty+(p_4(\omega)\Lambda_2(\gamma)+|\lambda_2|\Lambda_2(0))\|y-\tilde{y}\|_\infty.$$

Hence

$$d(N(x(\cdot,\omega),y(\cdot,\omega),\omega),N(\tilde{x}(\cdot,\omega),\tilde{y}(\cdot,\omega),\omega)) \leq \tilde{M}(\omega)d((x(\cdot,\omega),y(\cdot,\omega)),(\tilde{x}(\cdot,\omega),\tilde{y}(\cdot,\omega))),$$

where

$$d(x,y)=\begin{pmatrix}\|x(\cdot,\omega)-y(\cdot,\omega)\|_\infty\\ \|x(\cdot,\omega)-y(\cdot,\omega)\|_\infty\end{pmatrix}$$

and

$$\widetilde{M}(\omega)=\begin{pmatrix}\Lambda_1(\alpha)p_1(\omega)+|\lambda_1|\Lambda_1(0) & \Lambda_1(\alpha)p_2(\omega)\\ \Lambda_2(\gamma)p_3(\omega) & \Lambda_2(\gamma)p_4(\omega)+|\lambda_2|\Lambda_2(0)\end{pmatrix}.$$

Since for every $\omega\in\Omega, \widetilde{M}(\omega)\in M_{n\times n}(\mathbb{R}_+)$ converge to zero, then from Theorem 3.5 there exists a unique random solution of problem (1). This completes the proof.

Now, we present an existence result without Lipschitz conditions. We consider the following hypotheses:

(H_1) For every $\omega\in\Omega$, the functions $f(\cdot,\cdot,\cdot,\omega)$ and $g(\cdot,\cdot,\cdot,\omega)$ are continuous and $\omega\mapsto f(\cdot,\cdot,\cdot,\omega)$, $\omega\mapsto g(\cdot,\cdot,\cdot,\omega)$ are measurable.

(H_2) There exist measurable and bounded functions $\gamma_1,\gamma_2\colon\Omega\to\mathbb{R}_+$ such that

$$\|f(t,x,y,\omega)\|\leq\gamma_1(\omega)(\|x\|+\|y\|+1), \|g(t,x,y,\omega)\|\leq\gamma_2(\omega)(\|x\|+\|y\|+1)$$

for all $t\in[1,e]$, $\omega\in\Omega$ and $x,y\in\mathbb{R}^m$.

We use the Leray-Schauder random fixed point type theorem in a generalized Banach space to prove our existence result.

Theorem 4.2
Assume that the hypotheses (H_1), (H_2) *and condition*

$$\bigwedge\nolimits_1(\alpha)\gamma_1(\omega)+\bigwedge\nolimits_2(\gamma)\gamma_2(\omega)+|\lambda_1|\bigwedge\nolimits_1(0)+|\lambda_2|\bigwedge\nolimits_2(0)<1, \tag{2}$$

hold. Then the problem (1) *has a random solution defined on* $[1,e]$. *Moreover, the solution set* $S=\{(x,y)\colon\Omega\to C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m):\ (x(\cdot,\omega),y(\cdot,\omega)),\ \omega\in\Omega,$ *is a solution of* $(1)\}$ *is compact.*

Proof. Let $N\colon C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)\times\Omega\to C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$ be a random operator defined in Theorem 4.1. In order to apply theorem 3.6, we first show that N is completely continuous. The proof will be given in several steps.

STEP 1. $N(\cdot,\cdot,\omega)=(N_1(\cdot,\cdot,\omega),N_2(\cdot,\cdot,\omega))$ is continuous. Let (x_n,y_n) be a sequence such that $(x_n,y_n)\to(x,y)\in C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$ as $n\to\infty$. Since f is a continuous function, then

$$\begin{aligned}
&\sum_{j=1}^{n}|\phi_j|\|I^{\alpha+\beta+\gamma_j}f(\xi_j,x_n(\xi_j,\omega),y_n(\xi_j,\omega),\omega)\\
&\qquad\qquad -I^{\alpha+\beta+\gamma_j}f(\xi_j,x(\xi_j,\omega),y(\xi_j,\omega),\omega)\|_\infty\to 0 \qquad \text{as}\quad n\to\infty,\\
&\sum_{i=1}^{m}|\theta_i|\|I^{\alpha+\beta+\mu_i}f(\eta_i,x_n(\eta_i,\omega),y_n(\eta_i,\omega))\\
&\qquad\qquad -I^{\alpha+\beta+\mu_i}f(\eta_i,x(\eta_i,\omega),y(\eta_i,\omega))\|_\infty\to 0 \qquad \text{as}\qquad n\to\infty,\\
&\sum_{l=1}^{q}|v_l|\|I^{\alpha+\beta+\tau_l}f(\varphi_l,x_n(\varphi_l,\omega),y_n(\varphi_l,\omega))\\
&\qquad\qquad -I^{\alpha+\beta+\gamma_j}f(\varphi_l,x(\varphi_l,\omega),y(\varphi_l,\omega))\|_\infty\to 0 \qquad \text{as}\quad n\to\infty.
\end{aligned}$$

and

$$\begin{aligned}
&\|I^{\alpha+\beta}(f(t,x_n(t,\omega),y_n(t,\omega),\omega)-f(t,x(t,\omega),y(t,\omega),\omega)\|_\infty\\
&\qquad\qquad +|\lambda_1|\|I^{\beta}(x_n(t,\omega)-x(t,\omega)\|_\infty\to 0 \qquad \text{as}\quad n\to\infty
\end{aligned}$$

Thus

$$\|N_1(x_n(\cdot,\omega),y_n(\cdot,\omega),\omega)-N_1(x(\cdot,\omega),y(\cdot,\omega),\omega)\|_\infty\to 0 \qquad \text{as}\quad n\to\infty.$$

Similarly

$$\|N_2(x_n(\cdot,\omega),y_n(\cdot,\omega),\omega)-N_2(x(\cdot,\omega),y(.,\omega),\omega)\|_\infty\to 0 \qquad \text{as}\quad n\to\infty.$$

Therefore N is continuous.

STEP 2. We show that N maps bounded sets into bounded sets in $C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$. Indeed, it is enough to show that for any $q>0$ there exists a positive constant l such that for each $(x,y)\in B_q=\{(x,y)\in C([1,e],\mathbb{R})\times C([1,e],\mathbb{R}):\ (\|x\|_\infty,\|y\|_\infty)\le(q,q)\}$, we have $\|N(x,y,\omega)\|_\infty\le l=(l_1,l_2)$. Then for each $t\in[1,e]$ we get

$$\begin{aligned}
&\|N_1(x(t),y(t),\omega)\|\\
&\quad=\bigg\|\frac{1}{\Omega}\bigg[\Big(\Omega_4-\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\Big)\\
&\qquad\times\bigg(\sum_{j=1}^{n}\phi_j[I^{\alpha+\beta+\gamma_j}f(\xi_j,x(\xi_j,\omega),y(\xi_j,\omega))-\lambda_1 I^{\beta+\gamma_j}x(\xi_j,\omega)]
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{m} \theta_i [I^{\alpha+\beta+\mu_i} f(\eta_i, x(\eta_i,\omega), y(\eta_i,\omega)) - \lambda_1 I^{\beta+\mu_i} x(\eta_i,\omega)]\Big) \\
&+ \Big(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\Big) \\
&\times \Big(\sum_{l=1}^{q} \upsilon_l [I^{\alpha+\beta+\tau_l} f(\varphi_l, x(\varphi_l,\omega), y(\varphi_l,\omega)) - \lambda_1 I^{\beta+\tau_l} x(\varphi_l,\omega)] \\
&- \sum_{k=1}^{p} \varepsilon_k [I^{\alpha+\beta+\varsigma_k} f(\psi_k, x(\psi_k,\omega), y(\psi_k,\omega)) - \lambda_1 I^{\beta+\varsigma_k} x(\psi_k,\omega)]\Big)\Big] \\
&+ I^{\alpha+\beta} f(t, x(t,\omega), y(t,\omega)) - \lambda_1 I^{\beta} x(t,\omega)\Big\| \\
\le\ & \frac{1}{|\Omega|}\Big[\Big(|\Omega_4| + \frac{(\log t)^\beta}{\Gamma(\beta+1)}|\Omega_3|\Big) \\
&\times \Big(\sum_{j=1}^{n} |\phi_j| [I^{\alpha+\beta+\gamma_j} \|f(\xi_j, x(\xi_j,\omega), y(\xi_j,\omega))\| + |\lambda_1| I^{\beta+\gamma_j} \|x(\xi_j,\omega)\|] \\
&+ \sum_{i=1}^{m} |\theta_i| [I^{\alpha+\beta+\mu_i} \|f(\eta_i, x(\eta_i,\omega), y(\eta_i,\omega))\| + |\lambda_1| I^{\beta+\mu_i} \|x(\eta_i,\omega)\|]\Big) \\
&+ \Big(\frac{(\log t)^\beta}{\Gamma(\beta+1)}|\Omega_1| + |\Omega_2|\Big) \\
&\times \Big(\sum_{l=1}^{q} |\upsilon_l| [I^{\alpha+\beta+\tau_l} \|f(\varphi_l, x(\varphi_l,\omega), y(\varphi_l,\omega))\| + |\lambda_1| I^{\beta+\tau_l} \|x(\varphi_l,\omega)\|] \\
&+ \sum_{k=1}^{p} |\varepsilon_k| [I^{\alpha+\beta+\varsigma_k} \|f(\psi_k, x(\psi_k,\omega), y(\psi_k,\omega))\| + |\lambda_1| I^{\beta+\varsigma_k} \|x(\psi_k,\omega)\|]\Big)\Big] \\
&+ I^{\alpha+\beta} \|f(t, x(t,\omega), y(t,\omega))\| + |\lambda_1| I^{\beta} \|x(t,\omega)\| \\
\le\ & \frac{1}{|\Omega|}\Big[\Big(|\Omega_4| + \frac{(\log t)^\beta}{\Gamma(\beta+1)}|\Omega_3|\Big) \\
&\times \Big(\sum_{j=1}^{n} |\phi_j| [I^{\alpha+\beta+\gamma_j} \gamma_1(\omega)(\|x(\xi_j,\omega)\| + \|y(\xi_j,\omega)\|) |\lambda_1| I^{\beta+\gamma_j} \|x(\xi_j,\omega)\|] \\
&+ \sum_{i=1}^{m} |\theta_i| [I^{\alpha+\beta+\mu_i} \gamma_1(\omega)(\|x(\eta_i,\omega)\| + \|y(\eta_i,\omega)\|) + |\lambda_1| I^{\beta+\mu_i} \|x(\eta_i,\omega)\|]\Big) \\
&+ \Big(\frac{(\log t)^\beta}{\Gamma(\beta+1)}|\Omega_1| + |\Omega_2|\Big)
\end{aligned}
$$

$$\times\Big(\sum_{l=1}^{q}|\upsilon_l|[I^{\alpha+\beta+\tau_l}\gamma_1(\omega)(\|x(\varphi_l,\omega)\|+\|y(\varphi_l,\omega)\|)+|\lambda_1|I^{\beta+\tau_l}\|x(\varphi_l,\omega)\|]$$

$$+\sum_{k=1}^{p}|\varepsilon_k|[I^{\alpha+\beta+\varsigma_k}\gamma_1(\omega)(\|x(\psi_k,\omega)\|+\|y(\psi_k,\omega)\|)$$

$$+|\lambda_1|I^{\beta+\varsigma_k}\|x(\psi_k,\omega)\|]\Big)\Big]$$

$$+I^{\alpha+\beta}\gamma_1(\omega)(\|x(t,\omega)\|+\|y(t,\omega)\|)+|\lambda_1|I^{\beta}\|x(t,\omega)\|$$

$$\le \gamma_1(\omega)\Lambda_1(\alpha)(\|x\|+\|y\|)+|\lambda_1|q\Lambda_1(0)$$

$$\le \gamma_1(\omega)\Lambda_1(\alpha)(2q)+|\lambda_1|q\Lambda_1(0).$$

Then

$$\|N_1(x(\cdot,\omega),y(\cdot,\omega),\omega)\|_\infty \le \gamma_1(\omega)\Lambda_1(\alpha)(2q)+|\lambda|q\Lambda_1(0):=l_1.$$

Similarly, we have

$$\|N_2(x(\cdot,\omega),y(\cdot,\omega),\omega)\|_\infty \le \gamma_2(\omega)\Lambda_2(\gamma)(2q)+|\lambda|q\Lambda_2(0):=l_2.$$

Step 3. Next, we will show that N maps bounded sets into equicontinuous sets of $C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$. Let $B_r=\{(x,y)\in C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m):\|x\|\le r,\ \|y\|\le r\}$ be a bounded set in $C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$ as in Step 2. Let $t_1,t_2\in[1,e]$ with $t_1<t_2$ and $(x,y)\in B_r$. Then we have

$$\|N_1(x(t_2,\omega),y(t_2,\omega),\omega)-N_1(x(t_1,\omega),y(t_1,\omega),\omega)\|$$

$$=\Big\|\frac{1}{\Omega}\Big[\Big(\frac{(\log t_2)^\beta-(\log t_1)^\beta}{\Gamma(\beta+1)}\Omega_3\Big)$$

$$\times\Big(\sum_{j=1}^{n}\phi_j[I^{\alpha+\beta+\gamma_j}f(\xi_j,x(\xi_j,\omega),y(\xi_j,\omega),\omega)-\lambda_1I^{\beta+\gamma_j}x(\xi_j,\omega)]$$

$$-\sum_{i=1}^{m}\theta_i[I^{\alpha+\beta+\mu_i}f(\eta_i,x(\eta_i,\omega),y(\eta_i,\omega),\omega)-\lambda_1I^{\beta+\mu_i}x(\eta_i,\omega)]\Big)$$

$$+\Big(\frac{(\log t_2)^\beta-(\log t_1)^\beta}{\Gamma(\beta+1)}\Omega_1\Big)$$

$$\times\Big(\sum_{l=1}^{q}\upsilon_l[I^{\alpha+\beta+\tau_l}f(\varphi_l,x(\varphi_l,\omega),y(\varphi_l,\omega),\omega)-\lambda_1I^{\beta+\tau_l}x(\varphi_l,\omega)]$$

$$-\sum_{k=1}^{p}\varepsilon_k[I^{\alpha+\beta+\varsigma_k}f(\psi_k,x(\psi_k,\omega),y(\psi_k,\omega),\omega)-\lambda_1I^{\beta+\varsigma_k}x(\psi_k,\omega)]\Big)\Big]$$

$$+I^{\alpha+\beta}f(t,x(t_2,\omega),y(t_2,\omega),\omega)-\lambda_1I^{\beta}x(t_2,\omega)$$

$$
\begin{aligned}
& - I^{\alpha+\beta} f(t, x(t_1,\omega), y(t_1,\omega), \omega) + \lambda_1 I^{\beta} x(t_1,\omega)\Big\| \\
\leq\ & \frac{1}{|\Omega|}\Big[\Big(\frac{(\log t_2)^{\beta} - (\log t_1)^{\beta}}{\mathit{\Gamma}(\beta+1)}|\Omega_3|\Big) \\
& \times \Big(\sum_{j=1}^{n} |\phi_j| [I^{\alpha+\beta+\gamma_j}\gamma_1(\omega)(\|x(\xi_j,\omega)\| + \|y(\xi_j,\omega)\| + 1) \\
& + |\lambda_1| I^{\beta+\gamma_j} \|x(\xi_j,\omega)\|] \\
& + \sum_{i=1}^{m} |\theta_i| [I^{\alpha+\beta+\mu_i}\gamma_1(\omega)(\|x(\eta_i,\omega)\| + \|y(\eta_i,\omega)\| + 1) \\
& + |\lambda_1| I^{\beta+\mu_i} \|x(\eta_i,\omega)\|]\Big) \\
& + \Big(\frac{(\log t_2)^{\beta} - (\log t_1)^{\beta}}{\mathit{\Gamma}(\beta+1)}|\Omega_1|\Big) \\
& \times \Big(\sum_{l=1}^{q} |\upsilon_l| [I^{\alpha+\beta+\tau_l}\gamma_1(\omega)(\|x(\varphi_l,\omega)\| + \|y(\varphi_l,\omega)\| + 1) \\
& + |\lambda_1| I^{\beta+\tau_l} \|x(\varphi_l,\omega)\|] \\
& + \sum_{k=1}^{p} |\varepsilon_k| [I^{\alpha+\beta+\varsigma_k}\gamma_1(\omega)(\|x(\psi_k,\omega)\| + \|y(\psi_k,\omega)\|) \\
& + |\lambda_1| I^{\beta+\varsigma_k} \|x(\psi_k,\omega)\|]\Big)\Big] \\
& + \frac{|\lambda_1|\|x\|}{\Gamma(\beta+1)}\Big|(\log t_2)^{\beta} - (\log t_1)^{\beta} + 2(\log\frac{t_2}{t_1})^{\beta}\Big| \\
\leq\ & \frac{1}{|\Omega|}\Big[\Big(\frac{(\log t_2)^{\beta} - (\log t_1)^{\beta}}{\mathit{\Gamma}(\beta+1)}|\Omega_3|\Big) \\
& \times \Big(\sum_{j=1}^{n} |\phi_j| \Big[\frac{(2r+1)(\log \xi_j)^{\alpha+\beta+\gamma_j}\gamma_1(\omega)}{\Gamma(\alpha+\beta+\gamma_j+1)} + \frac{|\lambda_1| r (\log \xi_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)}\Big] \\
& + \sum_{i=1}^{m} |\theta_i| \Big[\frac{(2r+1)(\log \eta_i)^{\alpha+\beta+\mu_i}\gamma_1(\omega)}{\Gamma(\alpha+\beta+\mu_i+1)} + \frac{|\lambda_1| r (\log \eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)}\Big]\Big) \\
& + \Big(\frac{(\log t_2)^{\beta} - (\log t_1)^{\beta}}{\mathit{\Gamma}(\beta+1)}|\Omega_1|\Big) \\
& \times \Big(\sum_{l=1}^{q} |\upsilon_l| \Big[\frac{(2r+1)(\log \varphi_l)^{\alpha+\beta+\tau_l}\gamma_1(\omega)}{\Gamma(\alpha+\beta+\tau_l+1)} + \frac{|\lambda_1| r (\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)}\Big]
\end{aligned}
$$

$$+\sum_{k=1}^{p}|\varepsilon_k|\Big[\frac{(2r+1)(\log\psi_k)^{\alpha+\beta+\varsigma_k}\gamma_1(\omega)}{\Gamma(\alpha+\beta+\varsigma_k+1)}+\frac{|\lambda|r(\log\psi_k)^{\beta+\varsigma_k}}{\Gamma(\beta+\varsigma_k+1)}\Big]\Big)\Big]$$

$$+\frac{\gamma_1(\omega)(2r+1)}{\Gamma(\alpha+\beta+1)}\big|(\log t_2)^{\alpha+\beta}-(\log t_1)^{\alpha+\beta}\big|$$

$$+\frac{|\lambda_1|r}{\Gamma(\beta+1)}\big|(\log t_2)^{\beta}-(\log t_1)^{\beta}+2(\log\frac{t_2}{t_1})^{\beta}\big|.$$

Hence

$$\|N_1(x(t_2,\omega),y(t_2,\omega),\omega)-N_1(x(t_1,\omega),y(t_1,\omega),\omega)\|$$

$$\leq\frac{1}{|\Omega|}\Bigg[\Big(\frac{(\log t_2)^{\beta}-(\log t_1)^{\beta}}{\mathit{\Gamma}(\beta+1)}|\Omega_3|\Big)$$

$$\times\Bigg(\sum_{j=1}^{n}|\phi_j|\Big[\frac{(2r+1)(\log\xi_j)^{\alpha+\beta+\gamma_j}\gamma_1(\omega)}{\Gamma(\alpha+\beta+\gamma_j+1)}+\frac{|\lambda_1|r(\log\xi_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)}\Big]$$

$$+\sum_{i=1}^{m}|\theta_i|\Big[\frac{(2r+1)(\log\eta_i)^{\alpha+\beta+\mu_i}\gamma_1(\omega)}{\Gamma(\alpha+\beta+\mu_i+1)}+\frac{|\lambda_1|r(\log\eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)}\Big]\Bigg)$$

$$+\Big(\frac{(\log t_2)^{\beta}-(\log t_1)^{\beta}}{\mathit{\Gamma}(\beta+1)}|\Omega_1|\Big)$$

$$\times\Bigg(\sum_{l=1}^{q}|\upsilon_l|\Big[\frac{(2r+1)(\log\varphi_l)^{\alpha+\beta+\tau_l}\gamma_1(\omega)}{\Gamma(\alpha+\beta+\tau_l+1)}+\frac{|\lambda_1|r(\log\varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)}\Big]$$

$$+\sum_{k=1}^{p}|\varepsilon_k|\Big[\frac{(2r+1)(\log\psi_k)^{\alpha+\beta+\varsigma_k}\gamma_1(\omega)}{\Gamma(\alpha+\beta+\varsigma_k+1)}+\frac{|\lambda|r(\log\psi_k)^{\beta+\varsigma_k}}{\Gamma(\beta+\varsigma_k+1)}\Big]\Bigg)\Bigg]$$

$$+\frac{\gamma_1(\omega)(2r+1)}{\Gamma(\alpha+\beta+1)}\Big|(\log t_2)^{\alpha+\beta}-(\log t_1)^{\alpha+\beta}\Big|$$

$$+\frac{|\lambda_1|r}{\Gamma(\beta+1)}\Big|(\log t_2)^{\beta}-(\log t_1)^{\beta}+2(\log\frac{t_2}{t_1})^{\beta}\Big|$$

and

$$\|N_2(x(t_2,\omega),y(t_2,\omega),\omega)-N_2(x(t_1,\omega),y(t_1,\omega),\omega)\|$$

$$\leq\frac{1}{|\overline{\Omega}|}\Bigg[\Big(\frac{(\log t_2)^{\sigma}-(\log t_1)^{\sigma}}{\mathit{\Gamma}(\sigma+1)}|\overline{\Omega_3}|\Big)$$

$$\times\Bigg(\sum_{j=1}^{n}|\overline{\phi_j}|\Big[\frac{(2r+1)(\log\overline{\xi_j})^{\gamma+\sigma+\overline{\gamma_j}}\overline{\gamma_1}(\omega)}{\Gamma(\gamma+\sigma+\overline{\gamma_j}+1)}+\frac{|\lambda_2|r(\log\overline{\xi_j})^{\sigma+\overline{\gamma_j}}}{\Gamma(\sigma+\overline{\gamma_j}+1)}\Big]$$

$$+\sum_{i=1}^{m}|\overline{\theta_i}|\Big[\frac{(2r+1)(\log\overline{\eta_i})^{\gamma+\sigma+\overline{\mu_i}}\overline{\gamma_1}(\omega)}{\Gamma(\gamma+\sigma+\overline{\mu_i}+1)}+\frac{|\lambda_2|r(\log\overline{\eta_i})^{\sigma+\overline{\mu_i}}}{\Gamma(\sigma+\overline{\mu_i}+1)}\Big]\Bigg)$$

$$+\Big(\frac{(\log t_2)^{\sigma}-(\log t_1)^{\sigma}}{\Gamma(\sigma+1)}|\overline{\Omega_1}|\Big)$$

$$\times\Big(\sum_{l=1}^{q}|\overline{\upsilon_l}|\Big[\frac{2q(\log\overline{\varphi_l})^{\gamma+\sigma+\overline{\tau_l}}\overline{\gamma_1}(\omega)}{\Gamma(\gamma+\sigma+\overline{\tau_l}+1)}+\frac{|\lambda_2|r(\log\overline{\varphi_l})^{\sigma+\overline{\tau_l}}}{\Gamma(\sigma+\overline{\tau_l}+1)}\Big]$$

$$+\sum_{k=1}^{p}|\overline{\varepsilon_k}|\Big[\frac{(2r+1)(\log\overline{\psi_k})^{\gamma+\sigma+\overline{\varsigma_k}}\overline{\gamma_1}(\omega)}{\Gamma(\gamma+\sigma+\overline{\varsigma_k}+1)}+\frac{|\lambda_2|r(\log\overline{\psi_k})^{\sigma+\overline{\varsigma_k}}}{\Gamma(\sigma+\overline{\varsigma_k}+1)}\Big]\Big)\Big]$$

$$+\frac{\overline{\gamma}(\omega)(2r+1)}{\Gamma(\gamma+\sigma+1)}\Big|(\log t_2)^{\gamma+\sigma}-(\log t_1)^{\gamma+\sigma}\Big|$$

$$+\frac{|\lambda_2|r}{\Gamma(\sigma+1)}\Big|(\log t_2)^{\sigma}-(\log t_1)^{\sigma}+2(\log\frac{t_2}{t_1})^{\sigma}\Big|.$$

As $t_2-t_1\to 0$, the right-hand side of the above inequality tends to zero independently of $u\in B_q$. Therefore by the Arzela-Ascoli theorem the operator N is completely continuous.

STEP 4. It remains to show that $\mathcal{A}(\omega)$ is bounded, where

$$\mathcal{A}(\omega)=\{(x(\cdot,\omega),y(\cdot,\omega))\in C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m):$$
$$(x(\cdot,\omega),y(\cdot,\omega))=\lambda(\omega)N(x(\cdot,\omega),y(\cdot,\omega),\omega),\lambda(\omega)\in(0,1)\}.$$

Let $(x,y)\in\mathcal{A}(\omega)$. Then $x=\lambda(\omega)N_1(x,y)$ and $y=\lambda(\omega)N_2(x,y)$ for some $0<\lambda<1$. Thus, for $t\in[1,e]$, we have

$$\|x(t,\omega)\|\le\frac{1}{|\Omega|}\Big[\Big(|\Omega_4|+\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}|\Omega_3|\Big)$$

$$\times\Big(\sum_{j=1}^{n}|\phi_j|[I^{\alpha+\beta+\gamma_j}\|f(\xi_j,x(\xi_j,\omega),y(\xi_j,\omega))\|+|\lambda_1|I^{\beta+\gamma_j}\|x(\xi_j,\omega)\|]$$

$$+\sum_{i=1}^{m}|\theta_i|[I^{\alpha+\beta+\mu_i}\|f(\eta_i,x(\eta_i,\omega),y(\eta_i,\omega))\|+|\lambda_1|I^{\beta+\mu_i}\|x(\eta_i,\omega)\|]\Big)$$

$$+\Big(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}|\Omega_1|+|\Omega_2|\Big)$$

$$\times\Big(\sum_{l=1}^{q}|\upsilon_l|[I^{\alpha+\beta+\tau_l}\|f(\varphi_l,x(\varphi_l,\omega),y(\varphi_l,\omega))\|+|\lambda_1|I^{\beta+\tau_l}\|x(\varphi_l,\omega)\|]$$

$$+\sum_{k=1}^{p}|\varepsilon_k|[I^{\alpha+\beta+\varsigma_k}\|f(\psi_k,x(\psi_k,\omega),y(\psi_k,\omega))\|$$

$$+|\lambda_1|I^{\beta+\varsigma_k}\|x(\psi_k,\omega)\|]\Big)\Big]$$

$$+I^{\alpha+\beta}\|f(t,x(t,\omega),y(t,\omega))\|+|\lambda_1|I^{\beta}\|x(t,\omega)\|$$

$$
\begin{aligned}
&\leq \frac{1}{|\Omega|}\bigg[\Big(|\Omega_4| + \frac{(\log t)^\beta}{\Gamma(\beta+1)}|\Omega_3|\Big)\\
&\quad\times\bigg(\sum_{j=1}^{n}|\phi_j|[I^{\alpha+\beta+\gamma_j}\gamma_1(\omega)(\|x(\xi_j,\omega)\|+\|y(\xi_j,\omega)\|)\\
&\quad+|\lambda_1|I^{\beta+\gamma_j}\|x(\xi_j,\omega)\|]\\
&\quad+\sum_{i=1}^{m}|\theta_i|[I^{\alpha+\beta+\mu_i}\gamma_1(\omega)(\|x(\eta_i,\omega)\|+\|y(\eta_i,\omega)\|)\\
&\quad+|\lambda_1|I^{\beta+\mu_i}\|x(\eta_i,\omega)\|]\bigg)\\
&\quad+\Big(\frac{(\log t)^\beta}{\Gamma(\beta+1)}|\Omega_1|+|\Omega_2|\Big)\\
&\quad\times\bigg(\sum_{l=1}^{q}|\upsilon_l|[I^{\alpha+\beta+\tau_l}\gamma_1(\omega)(\|x(\varphi_l,\omega)\|+\|y(\varphi_l,\omega)\|)\\
&\quad+|\lambda_1|I^{\beta+\tau_l}\|x(\varphi_l,\omega)\|]\\
&\quad+\sum_{k=1}^{p}|\varepsilon_k|[I^{\alpha+\beta+\varsigma_k}\gamma_1(\omega)(\|x(\psi_k,\omega)\|+\|y(\psi_k,\omega)\|)\\
&\quad+|\lambda_1|I^{\beta+\varsigma_k}\|x(\psi_k,\omega)\|]\bigg)\bigg]\\
&\quad+I^{\alpha+\beta}\gamma_1(\omega)(\|x(t,\omega)\|+\|y(t,\omega)\|)+|\lambda_1|I^{\beta}\|x(t,\omega)\|
\end{aligned}
$$

Then

$$
\begin{aligned}
\|x(t,\omega)\| \leq{}& \gamma_1(\omega)(\|x(\cdot,\omega)\|_\infty+\|y(\cdot,\omega)\|_\infty+1)\bigwedge\nolimits_1(\alpha)\\
&+|\lambda_1|\bigwedge_1(0)(\|x(\cdot,\omega)\|_\infty+\|y(\cdot,\omega)\|_\infty)
\end{aligned}
$$

We have also

$$
\begin{aligned}
\|y(t,\omega)\| \leq{}& \gamma_2(\omega)(\|x(\cdot,\omega)\|_\infty+\|y(\cdot,\omega)\|_\infty+1)\bigwedge\nolimits_2(\gamma)\\
&+|\lambda_2|\bigwedge_2(0)(\|x(\cdot,\omega)\|_\infty+\|y(\cdot,\omega)\|_\infty)
\end{aligned}
$$

Therefore

$$
\|x(t,\omega)\|+\|y(t,\omega)\| \leq C+K(\|x(\cdot,\omega)\|_\infty+\|y(\cdot,\omega)\|_\infty),
$$

where

$$C=\gamma_1(\omega)+\gamma_2(\omega),\ K=\bigwedge\nolimits_1(\alpha)\gamma_1(\omega)+\bigwedge\nolimits_2(\gamma)\gamma_2(\omega)+|\lambda_1|\bigwedge\nolimits_1(0)+|\lambda_2|\bigwedge\nolimits_2(0).$$

Hence, from (2), we get

$$\|x(\cdot,\omega)\|_\infty+\|y(\cdot,\omega)\|_\infty\le\frac{\gamma_1(\omega)+\gamma(\omega)_2}{1-K}:=K_*.$$

Consequently, $\|x\|\le K_*$ and $\|y\|\le K_*$. This shows that $\mathcal{A}(\omega)$ is bounded. As a consequence of Theorem 3.6 we deduce that N has a random fixed point $\omega\to(x(\cdot,\omega),y(\cdot,\omega))$ which is a solution to the problem (1).

Step 5. Compactness of the solution set. Let $\{(x_n,y_n)\}_{n\in\mathbb{N}}\subset S$ be a sequence. For every $n\in\mathbb{N}$ and for fixe $\omega\in\Omega$, we get

$$\begin{aligned}x_n(t,\omega)=\frac{1}{\Omega}\Bigg[&\Big(\Omega_4-\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\Big)\\
&\times\Bigg(\sum_{j=1}^{n}\phi_j[I^{\alpha+\beta+\gamma_j}f(\xi_j,x_n(\xi_j,\omega),y_n(\xi_j,\omega),\omega)-\lambda_1I^{\beta+\gamma_j}x_n(\xi_j,\omega)]\\
&-\sum_{i=1}^{m}\theta_i[I^{\alpha+\beta+\mu_i}f(\eta_i,x_n(\eta_i,\omega),y_n(\eta_i,\omega),\omega)-\lambda_1I^{\beta+\mu_i}x_n(\eta_i,\omega)]\Bigg)\\
&+(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1-\Omega_2)\\
&\times\Bigg(\sum_{l=1}^{q}\upsilon_l[I^{\alpha+\beta+\tau_l}f(\varphi_l,x_n(\varphi_l,\omega),y_n(\varphi_l,\omega),\omega)-\lambda_1I^{\beta+\tau_l}x_n(\varphi_l,\omega)]\\
&-\sum_{k=1}^{p}\varepsilon_k[I^{\alpha+\beta+\varsigma_k}f(\psi_k,x_n(\psi_k,\omega),y_n(\psi_k,\omega),\omega)\\
&-\lambda_1I^{\beta+\varsigma_k}x_n(\psi_k,\omega)]\Bigg)\Bigg]\\
&+I^{\alpha+\beta}f(t,x_n(t,\omega),y_n(t,\omega),\omega)-\lambda_1I^\beta x_n(t,\omega)\end{aligned}$$

and

$$\begin{aligned}y_n(t,\omega)=\frac{1}{\overline{\Omega}}\Bigg[&\Big(\overline{\Omega_4}-\frac{(\log t)^\sigma}{\Gamma(\sigma+1)}\overline{\Omega_3}\Big)\\
&\times\Bigg(\sum_{j=1}^{n}\overline{\phi_j}[I^{\gamma+\sigma+\overline{\gamma_j}}g(\overline{\xi_j},x_n(\overline{\xi_j},\omega),y_n(\overline{\xi_j},\omega),\omega)-\lambda_2I^{\sigma+\overline{\gamma_j}}y_n(\overline{\xi_j},\omega)]\\
&-\sum_{i=1}^{m}\overline{\theta_i}[I^{\gamma+\sigma+\overline{\mu_i}}g(\overline{\eta_i},x_n(\overline{\eta_i},\omega),y_n(\overline{\eta_i},\omega),\omega)-\lambda_2I^{\sigma+\overline{\mu_i}}y_n(\overline{\eta_i},\omega)]\Bigg)\end{aligned}$$

$$+\Big(\frac{(\log t)^{\sigma}}{\Gamma(\sigma+1)}\overline{\Omega_1}-\overline{\Omega_2}\Big)$$

$$\times\Big(\sum_{l=1}^{q}\overline{\upsilon_l}[I^{\gamma+\sigma+\overline{\tau_l}}g(\overline{\varphi_l},x_n(\overline{\varphi_l},\omega),y_n(\overline{\varphi_l},\omega),\omega)-\lambda_2 I^{\sigma+\overline{\tau_l}}y_n(\overline{\varphi_l},\omega)]$$

$$-\sum_{k=1}^{p}\overline{\varepsilon_k}[I^{\gamma+\sigma+\overline{\varsigma_k}}g(\overline{\psi_k},x_n(\overline{\psi_k},\omega),y_n(\overline{\psi_k},\omega),\omega)$$

$$-\lambda_2 I^{\sigma+\overline{\varsigma_k}}y_n(\overline{\psi_k},\omega)]\Big)\Big]$$

$$+I^{\gamma+\sigma}g(t,x_n(t,\omega),y_n(t,\omega),\omega)-\lambda_2 I^{\sigma}y_n(t,\omega).$$

As in Steps 3, 4, we can prove that subsequence $\{(x_{nk},y_{nk})\}_{k\in\mathbb{N}}$ of $\{(x_n,y_n)\}_{n\in\mathbb{N}}$ converge to some $(x(\cdot,\omega),y(\cdot,\omega))\in C([1,e],\mathbb{R}^m)\times C([1,e],\mathbb{R}^m)$, such that

$$\omega\to x(t,\omega),\quad \omega\to y(\cdot,\omega)$$

are measurable functions. Since $f(\cdot,\cdot,\cdot,\omega)$ and $g(\cdot,\cdot,\cdot,\omega)$ are continuous functions, then

$$x(t,\omega)=\frac{1}{\Omega}\Big[\Big(\Omega_4-\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\Big)$$

$$\times\Big(\sum_{j=1}^{n}\phi_j[I^{\alpha+\beta+\gamma_j}f(\xi_j,x(\xi_j,\omega),y(\xi_j,\omega),\omega)-\lambda_1 I^{\beta+\gamma_j}x(\xi_j,\omega)]$$

$$-\sum_{i=1}^{m}\theta_i[I^{\alpha+\beta+\mu_i}f(\eta_i,x(\eta_i,\omega),y(\eta_i,\omega),\omega)-\lambda_1 I^{\beta+\mu_i}x(\eta_i,\omega)]\Big)$$

$$+\Big(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1-\Omega_2\Big)$$

$$\times\Big(\sum_{l=1}^{q}\upsilon_l[I^{\alpha+\beta+\tau_l}f(\varphi_l,x(\varphi_l,\omega),y(\varphi_l,\omega),\omega)-\lambda_1 I^{\beta+\tau_l}x(\varphi_l,\omega)]$$

$$-\sum_{k=1}^{p}\varepsilon_k[I^{\alpha+\beta+\varsigma_k}f(\psi_k,x(\psi_k,\omega),y(\psi_k,\omega),\omega)-\lambda_1 I^{\beta+\varsigma_k}x(\psi_k,\omega)]\Big)\Big]$$

$$+I^{\alpha+\beta}f(t,x(t,\omega),y(t,\omega),\omega)-\lambda_1 I^{\beta}x(t,\omega)$$

and

$$y(t,\omega)=\frac{1}{\overline{\Omega}}\Big[\Big(\overline{\Omega_4}-\frac{(\log t)^{\sigma}}{\Gamma(\sigma+1)}\overline{\Omega_3}\Big)$$

$$\times\Big(\sum_{j=1}^{n}\overline{\phi_j}[I^{\gamma+\sigma+\overline{\gamma_j}}g(\overline{\xi_j},x(\overline{\xi_j},\omega),y(\overline{\xi_j},\omega),\omega)-\lambda_2 I^{\sigma+\overline{\gamma_j}}y(\overline{\xi_j},\omega)]$$

$$- \sum_{i=1}^{m} \overline{\theta_i}[I^{\gamma+\sigma+\overline{\mu_i}} g(\overline{\eta_i}, x(\overline{\eta_i}, \omega), y(\overline{\eta_i}, \omega), \omega) - \lambda_2 I^{\sigma+\overline{\mu_i}} y(\overline{\eta_i}, \omega)]\Big)$$

$$+ \Big(\frac{(\log t)^\sigma}{\Gamma(\sigma+1)} \overline{\Omega_1} - \overline{\Omega_2}\Big)$$

$$\times \Big(\sum_{l=1}^{q} \overline{v_l}[I^{\gamma+\sigma+\overline{\tau_l}} g(\overline{\varphi_l}, x(\overline{\varphi_l}, \omega), y(\overline{\varphi_l}, \omega), \omega) - \lambda_2 I^{\sigma+\overline{\tau_l}} y(\overline{\varphi_l}, \omega)]$$

$$- \sum_{k=1}^{p} \overline{\varepsilon_k}[I^{\gamma+\sigma+\overline{\varsigma_k}} g(\overline{\psi_k}, x(\overline{\psi_k}, \omega), y(\overline{\psi_k}, \omega), \omega) - \lambda_2 I^{\sigma+\overline{\varsigma_k}} y(\overline{\psi_k}, \omega)]\Big)\Big]$$

$$+ I^{\gamma+\sigma} g(t, x(t,\omega), y(t,\omega), \omega) - \lambda_2 I^\sigma y(t,\omega).$$

So S is compact.

5. Examples

In this section we consider two examples for illustrate our main results.

EXAMPLE 5.1
Consider the following system of fractional differential equation:

$$\begin{cases} D^{\frac{1}{2}}(D^{\frac{2}{3}} + \lambda_1)x(t,\omega) = f(t, x(t,\omega), y(t,\omega), \omega) \\ D^{\frac{2}{3}}(D^{\frac{2}{5}} + \lambda_2)y(t,\omega) = g(t, x(t,\omega), y(t,\omega), \omega) \\ 4I^{\frac{2}{3}} x(\frac{2e}{3}, \omega) + I^{\frac{3}{5}} x(\frac{e+1}{3}, \omega) = \frac{2}{5} I^{\frac{2}{3}} x(\frac{e+2}{3}, \omega) \\ \frac{2}{3} I^{\frac{1}{5}} x(\frac{e}{2}, \omega) = I^{\frac{2}{5}} x(\frac{2e}{5}, \omega) + 3I^{\frac{1}{4}} x(\frac{4e}{5}, \omega) \\ \frac{2}{5} I^{\frac{2}{3}} y(\frac{e+2}{4}, \omega) + \frac{1}{2} I^{\frac{1}{2}} y(\frac{e+3}{4}, \omega) + \frac{4}{5} I^{\frac{3}{2}} y(\frac{e+4}{4}, \omega) = 4I^{\frac{2}{3}} y(\frac{3e}{4}, \omega) \\ \frac{5}{7} I^{\frac{1}{2}} y(\frac{3e}{5}, \omega) = \frac{1}{5} I^{\frac{3}{4}} y(\frac{4e}{5}, \omega) \end{cases} \tag{3}$$

where $\alpha = \frac{1}{2}$, $\beta = \frac{2}{3}$, $\gamma = \frac{2}{3}$, $\sigma = \frac{2}{5}$, $\lambda_1 = \frac{1}{6\Lambda_1(0)}$, $\lambda_2 = \frac{1}{6\Lambda_2(0)}$, $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra, $f, g\colon [1,e] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ defined by

$$f(t,x,y,\omega) = \frac{\cos(x+y)}{6\Lambda_1(\alpha)} + \omega t, \quad g(t,x,y,\omega) = \frac{|x+y|}{6\Lambda_2(\gamma)} + \frac{\log t}{t} + \omega^2,$$

where

$$\Lambda_1(\alpha) \approx 101,544, \quad \Lambda_1(0) \approx 175,398$$

and

$$\Lambda_2(\gamma) \approx 6,598, \quad \Lambda_2(0) \approx 15,945.$$

We can easily show that

$$|f(t,x,y,\omega) - f(t,x,y,\omega)| \le \frac{1}{6\Lambda_1(\alpha)}(|x-\bar{x}| + |y-\bar{y}|) \quad \text{for all } x, \bar{x}, y, \bar{y} \in \mathbb{R},\ t \in [1,e]$$

and

$$|g(t,x,y,\omega)-g(t,x,y,\omega)| \leq \frac{1}{6\Lambda_2(\alpha)}(|x-\bar{x}|+|y-\bar{y}|) \quad \text{for all } x,\bar{x},y,\bar{y}\in\mathbb{R},\ t\in[1,e].$$

Hence

$$\widetilde{M}(\omega) = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix}, \qquad \det(M-\lambda I) = \Big(\lambda-\frac{1}{2}\Big)\Big(\lambda-\frac{1}{6}\Big).$$

We observe that $|\rho(M(\omega))| = \frac{1}{2} < 1$, then $M(\omega)$ converge to 0. Therefore, all the conditions of theorem 4.1 are satisfied. Hence the problem (3) has a unique random solution.

Example 5.2

Consider the following system of fractional differential equation:

$$\begin{cases} D^{\frac{1}{2}}(D^{\frac{2}{3}}+\lambda_1)x(t,\omega) = f(t,x(t,\omega),y(t,\omega),\omega) \\ D^{\frac{2}{3}}(D^{\frac{2}{5}}+\lambda_2)y(t,\omega) = g(t,x(t,\omega),y(t,\omega),\omega) \\ 4I^{\frac{2}{3}}x(\frac{2e}{3},\omega) + I^{\frac{3}{5}}x(\frac{e+1}{3},\omega) = \frac{2}{5}I^{\frac{2}{3}}x(\frac{e+2}{3},\omega) \\ \frac{2}{3}I^{\frac{1}{5}}x(\frac{e}{2},\omega) = I^{\frac{2}{5}}x(\frac{2e}{5},\omega) + 3I^{\frac{1}{4}}x(\frac{4e}{5},\omega) \\ \frac{2}{5}I^{\frac{2}{3}}y(\frac{e+2}{4},\omega) + \frac{1}{2}I^{\frac{1}{2}}y(\frac{e+3}{4},\omega) + \frac{4}{5}I^{\frac{3}{2}}y(\frac{e+4}{4},\omega) = 4I^{\frac{2}{3}}y(\frac{3e}{4},\omega) \\ \frac{5}{7}I^{\frac{1}{2}}y(\frac{3e}{5},\omega) = \frac{1}{5}I^{\frac{3}{4}}y(\frac{4e}{5},\omega) \end{cases} \tag{4}$$

where $\alpha = \frac{1}{2}$, $\beta = \frac{2}{3}$, $\gamma = \frac{2}{3}$, $\sigma = \frac{2}{5}$, $\lambda_1 = \frac{1}{6\Lambda_1(0)}$, $\lambda_2 = \frac{1}{6\Lambda_2(0)}$. Here

$$f(t,x,y,\omega) = \frac{t\omega^2x^2}{2(1+\omega^2)(1+x^2+y^2)}$$

and

$$g(t,x,y,\omega) = \frac{t\omega^2y^2}{2(1+\omega^2)(1+x^2+y^2)}.$$

Clearly, the map $(t,\omega)\mapsto f(t,x,y,\omega)$ is jointly continuous for all $x,y\in\mathbb{R}$. Thus the functions f and g are Carathéodory on $[1,e]\times\mathbb{R}\times\mathbb{R}\times\mathcal{F}$. Firstly, we show that f and g are Lipschitz functions. Then

$$|f(t,x,y,\omega)| \leq \frac{\omega^2}{6\bigwedge_1(\alpha)(1+\omega^2)} \qquad \text{for all } x,y\in\mathbb{R}$$

and

$$|g(t,x,y,\omega)| \leq \frac{\omega^2}{6\bigwedge_2(\gamma)(1+\omega^2)} \qquad \text{for all } x,y\in\mathbb{R}.$$

Therefore, all the conditions of Theorem 4.2 hold. Then the problem (4) has at least one random solution.

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Zakaria Malki
Farida Berhoun
Abdelghani Ouahab
Laboratoire de Mathématiques
Université de Sidi Bel-Abbès
B.P. 89 22000
Sidi Bel-Abbès
Algérie
E-mail: malkizaki@hotmail.com
berhoun22@yahoo.fr
agh_ouahab@yahoo.fr

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