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Global existence and blow-up of generalized self-similar solutions for a space-fractional diffusion equation with mixed conditions

Abstract. This paper investigates the problem of the existence and uniqueness of solutions under the generalized self-similar forms to the space-fractional diffusion equation. Therefore, through applying the properties of Schauder's and Banach's fixed point theorems; we establish several results on the global existence and blow-up of generalized self-similar solutions to this equation.

1. Introduction

The partial differential equations (PDEs) of fractional order appear as a natural description of observed evolution phenomena in various scientific areas. The fractional derivative operators are non-local and this property is important in application because it allows modelling the dynamics of many problems in physics, chemistry, engineering, medicine, economics, control theory, etc. For further reading on the subject, readers can refer to the following books (Samko et al. 1993 [16], Podlubny 1999 [15], Kilbas et al. 2006 [9], Diethelm 2010 [7]).

In this work, we shall give an example of a class of well-known fractional-order's partial differential equations (PDEs), which allow to describe the diffusion phenomena; it is a space-fractional diffusion equation and is written as follows

$$\frac{\partial u}{\partial t} = \frac{\partial^\alpha u}{\partial x^\alpha} + \mu u, \qquad 1 < \alpha \le 2,\ \mu \in \mathbb{R} \tag{1}$$

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with

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \begin{cases} \dfrac{\partial^n u}{\partial x^n}, & \alpha = n \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x (x-s)^{n-\alpha-1} \frac{\partial^n}{\partial s^n} u(s,t) ds, & n-1 < \alpha < n \in \mathbb{N}, \end{cases}$$

where $u = u(x,t)$ is a scalar function of space variable $x \in [0, X]$, $X > 0$ and time $t \in [0, T)$ with a finite or infinite positive constant T.

The existence and uniqueness of solutions for fractional differential equations or fractional-order's PDEs have been investigated in recent years. For more on the subject, we refer the reader to the following works [1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 17].

Recently, the Lie group analysis of this equation has been discussed by Luchko et al. (see [6, 10]), which studied the equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = d \frac{\partial^\beta u}{\partial x^\beta}, \qquad x > 0,\ t > 0,\ d > 0,\ \alpha, \beta \geq 0, \tag{2}$$

to obtain the partial scale-invariant solutions of this equation.

Hundreds of years ago, Sophus Lie initiated working on the method of differential equations' group analysis. A symmetry group of a system of differential equations can in a way be said to be a group that transforms solutions of the system to other solutions. Special types of invariant solutions under a subgroup of the full symmetry group of the system can be determined for partial differential equations. Such "group-invariant" can be found if we solve a reduced system of equations with fewer independent variables than the original system.

In [6], the scale-invariant solutions of the time-fractional diffusion equation ($\beta = 2$ in (2)) were found using this method. Considering equation (2), an ordinary differential equation of fractional order with a new independent variable $\eta = xt^{-\alpha/\beta}$ is solved in order to find the scale-invariant solutions. The derivatives there are the Erdélyi-Kober derivatives (left- and right-hand sided). Hence, they depend on the parameters α, β of equation (2) and on a parameter γ of its scaling group. The general solution of this differential equation of fractional order is obtained in terms of the generalized Wright function.

For $\mu = 0$, the equation becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^\alpha u}{\partial x^\alpha}, \qquad 1 < \alpha \leq 2. \tag{3}$$

In [4], Basti et al. applied the Banach contraction principle, Schauder fixed-point theorem and the nonlinear alternative of Leray-Schauder type, to show the existence and uniqueness of self-similar solutions for the space-fractional heat equation (3). The proposed solution was

$$u(x,t) = t^\beta f\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) \qquad \text{with } (x,t) \in [0,X] \times [t_0, \infty),$$

where $X, t_0 > 0$, f is the basic profile and $\beta \in \mathbb{R}$ is a constant chosen so that the solutions exist.

Our main goal in this work is to determine the existence, uniqueness and main properties of the global or blow-up solution in time of the space-fractional PDE (1), under the generalized self-similar form which is

$$u(x,t) = c(t) f\Big(\frac{x}{a(t)}\Big) \qquad \text{with } a, c \in \mathbb{R}_+^*.$$

The functions $a(t)$ and $c(t)$ which depend on time t and the basic profile f are not known in advance and are to be identified.

2. Definitions and preliminary results

We present in this section some necessary definitions. We denote by $C[0,\lambda]$ the Banach space of continuous functions from $[0,\lambda]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup_{0\le\eta\le\lambda} |y(\eta)|.$$

We start with the definitions introduced in [9] with a slight modification in the notation.

Definition 1 ([9])
The left-sided (arbitrary) fractional integral of order $\alpha > 0$ of a continuous function $y\colon [0,\lambda] \to \mathbb{R}$ is given by

$$\mathcal{I}_{0^+}^{\alpha} y(\eta) = \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-\xi)^{\alpha-1} y(\xi) d\xi, \qquad \eta \in [0,\lambda].$$

Definition 2 (Caputo fractional derivative [9])
The left-sided Caputo fractional derivative of order $\alpha > 0$ of a function $y\colon [0,\lambda] \to \mathbb{R}$ is given by

$${}^C\mathcal{D}_{0^+}^{\alpha} y(\eta) = \mathcal{I}_{0^+}^{n-\alpha} \frac{d^n y(\eta)}{d\eta^n} = \frac{1}{\Gamma(n-\alpha)} \int_0^\eta (\eta-\xi)^{n-\alpha-1} \frac{d^n y(\xi)}{d\xi^n} d\xi, \qquad n = [\alpha]+1.$$

Lemma 3 ([9])
Assume that ${}^C\mathcal{D}_{0^+}^{\alpha} y \in C[0,\lambda]$ *for all* $\alpha > 0$ *then*

$$\mathcal{I}_{0^+}^{\alpha}\, {}^C\mathcal{D}_{0^+}^{\alpha} y(\eta) = y(\eta) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} \eta^k,$$

where $n = [\alpha]+1$.

Remark 4 ([4])
For all $y, {}^C\mathcal{D}_{0^+}^{\alpha} y \in C[0,\lambda]$, where $1 < \alpha \le 2$, we have

$$\mathcal{I}_{0^+}^{\alpha-1}\, {}^C\mathcal{D}_{0^+}^{\alpha} y(\eta) = y'(\eta) - y'(0).$$

Moreover; if $y'(0) = 0$, then we have for any $\eta \in [0,\lambda]$,

$$|y'(\eta)| \le \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} \|{}^C\mathcal{D}_{0^+}^{\alpha} y\|_\infty. \tag{4}$$

Lemma 5 ([4])
We define

$$\Omega = \{y \in C[0,\lambda] : \; y'(0) = 0\}. \tag{5}$$

Then $(\Omega, \|\cdot\|_\infty)$ *is a Banach space.*

Lemma 6 ([4])
Let $1 < \alpha \le 2$ *and* $\mathcal{A}\colon \Omega \to C[0,\lambda]$ *be an integral operator, defined by*

$$\mathcal{A}y(\eta) = U + \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1}\Big(\beta y(\xi) - \frac{\xi}{\alpha} y'(\xi)\Big) d\xi,$$

equipped with the standard norm

$$\|\mathcal{A}y\|_\infty = \sup_{0\le\eta\le\lambda} |\mathcal{A}y(\eta)|.$$

Then $\mathcal{A}(\Omega) \subset \Omega$.

Theorem 7 (Banach's fixed point [8])
Let Ω *be a non-empty closed subset of a Banach space* P*, then any contraction mapping* $\mathcal{A}$ *of* Ω *into itself has a unique fixed point.*

Theorem 8 (Schauder's fixed point [8])
Let P *be a Banach space, and* Ω *be a closed, convex and nonempty subset of* P*. Let* $\mathcal{A}\colon \Omega \to \Omega$ *be a continuous mapping such that* $\mathcal{A}(\Omega)$ *is a relatively compact subset of* P*. Then* $\mathcal{A}$ *has at least one fixed point in* Ω*.*

3. Main results

3.1. Statement of the problem

In this part, we first attempt to find the equivalent approximate to the following problem of the space-fractional diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^\alpha u}{\partial x^\alpha} + \mu u, \qquad (x,t) \in [0,X] \times [0,\infty), \; \mu \in \mathbb{R}, \; 1 < \alpha \le 2, \\ u(0,t) = c(t)U, \; \dfrac{\partial u}{\partial x}(0,t) = 0, \; u(x,0) = f(x), \qquad U \in \mathbb{R}. \end{cases}$$

Under the generalized self-similar form which is

$$u(x,t) = c(t) f(\eta) \qquad \text{with } \eta = \frac{x}{a(t)} \text{ and } a, c \in \mathbb{R}_+^*, \tag{6}$$

where

$$a(0) = c(0) = 1.$$

We should first deduce the equation satisfied by the function f in (6) used for the definition of self-similar solutions.

Theorem 9
Let $1 < \alpha \leq 2$ *and* $(x,t) \in [0,X]\times[0,\infty)$ *with* $0 < X \leq \lambda a(t)$ *for some* $\lambda > 0$. *Then the transformation* (6) *reduces the partial differential equation of space-fractional order* (1) *to the ordinary differential equation of fractional order of the form*

$$^{C}\mathcal{D}^{\alpha}_{0+} f(\eta) = \beta f(\eta) + \gamma\eta f'(\eta), \qquad \beta,\gamma \in \mathbb{R},\ \eta \in [0,\lambda],$$

where

$$\begin{cases} \dot{c}(t) = c(t)(\beta a^{-\alpha}(t) + \mu), \\ \dot{a}(t) = -\gamma a^{1-\alpha}(t). \end{cases} \tag{7}$$

Proof. The fractional equation resulting from the substitution of expression (6) in the original PDE (1), should be reduced to the standard bilinear functional equation. First, for $\eta = \frac{x}{a(t)}$, we get $\eta \in [0,\lambda]$ and

$$\frac{\partial u}{\partial t} = \dot{c}(t) f(\eta) - c(t)\frac{\dot{a}(t)}{a(t)}\eta f'(\eta). \tag{8}$$

On the other hand, we get for $\xi = \frac{s}{a(t)}$, that

$$\begin{aligned} \frac{\partial^{\alpha} u}{\partial x^{\alpha}} &= \frac{c(t)}{\Gamma(n-\alpha)}\int_0^x (x-s)^{n-\alpha-1}\frac{d^n}{ds^n} f\Big(\frac{s}{a(t)}\Big)ds \\ &= \frac{a(t)c(t)}{\Gamma(n-\alpha)}\int_0^{\eta} a^{(n-\alpha-1)}(t)(\eta-\xi)^{n-\alpha-1}\frac{d^n}{a^n(t)d\xi^n} f(\xi)d\xi \\ &= \frac{c(t)}{a^{\alpha}(t)}\,{}^{C}\mathcal{D}^{\alpha}_{0+} f(\eta). \end{aligned} \tag{9}$$

If we replace (8) and (9) in (1), we obtain the following equation

$$^{C}\mathcal{D}^{\alpha}_{0+} f(\eta) = a^{\alpha}(t)\Big(\frac{\dot{c}(t)}{c(t)} - \mu\Big) f(\eta) - a^{\alpha-1}(t)\dot{a}(t)\eta f'(\eta).$$

By choosing

$$\beta = a^{\alpha}(t)\Big(\frac{\dot{c}(t)}{c(t)} - \mu\Big) \quad \text{and} \quad \gamma = -a^{\alpha-1}(t)\dot{a}(t),$$

we get the coupled system (7) and

$$^{C}\mathcal{D}^{\alpha}_{0+} f(\eta) = \beta f(\eta) + \gamma\eta f'(\eta), \qquad \eta \in [0,\lambda]\,.$$

The proof is completed.

3.2. Existence and uniqueness results of the basic profile

According to the preceding part, Theorem 9, we study this problem

$$^{C}\mathcal{D}^{\alpha}_{0+} f(\eta) = \beta f(\eta) + \gamma\eta f'(\eta), \qquad 1 < \alpha \leq 2,\ \eta \in [0,\lambda], \tag{10}$$

in which $\lambda > 0$ is an arbitrary real constant with the conditions

$$f(0) = U, \quad f'(0) = 0. \tag{11}$$

Lemma 10
Let $\alpha,\beta,\gamma,\lambda\in\mathbb{R}$, be such that $1<\alpha\le 2$ and $\lambda>0$. We give $f, f', {}^{C}\mathcal{D}^{\alpha}_{0+}f\in C[0,\lambda]$. Then the problem (10)–(11) is equivalent to the integral equation

$$f(\eta)=U+\frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\xi)^{\alpha-1}(\beta f(\xi)+\gamma\xi f'(\xi))d\xi \qquad \textit{for all } \eta\in[0,\lambda].$$

Proof. Let $\alpha,\beta,\gamma,\lambda\in\mathbb{R}$, be such that $1<\alpha\le 2$ and $\lambda>0$. We may apply Lemma 3 to reduce the fractional equation (10) to an equivalent fractional integral equation. By applying $\mathcal{I}^{\alpha}_{0+}$ to equation (10) we obtain

$$\mathcal{I}^{\alpha}_{0+}\,{}^{C}\mathcal{D}^{\alpha}_{0+}f(\eta)=\mathcal{I}^{\alpha}_{0+}(\beta f(\eta)+\gamma\eta f'(\eta)). \tag{12}$$

From Lemma 3, we find easily

$$\mathcal{I}^{\alpha}_{0+}\,{}^{C}\mathcal{D}^{\alpha}_{0+}f(\eta)=f(\eta)-f(0)-\eta f'(0).$$

Then, the fractional integral equation (12), gives

$$f(\eta)=\mathcal{I}^{\alpha}_{0+}(\beta f(\eta)+\gamma\eta f'(\eta))+f(0)+\eta f'(0). \tag{13}$$

Using (11) in (13), we find that (10)–(11) is equivalent to

$$f(\eta)=U+\frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\xi)^{\alpha-1}(\beta f(\xi)+\gamma\xi f'(\xi))d\xi.$$

The proof is completed.

Lemma 11
Let $\mathcal{A}\colon\Omega\to C[0,\lambda]$ be an integral operator, which is defined by

$$\mathcal{A}f(\eta)=U+\frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\xi)^{\alpha-1}(\beta f(\xi)+\gamma\xi f'(\xi))d\xi,$$

equipped with the standard norm

$$\|\mathcal{A}f\|_{\infty}=\sup_{0\le\eta\le\lambda}|\mathcal{A}f(\eta)|.$$

Then $\mathcal{A}(\Omega)\subset\Omega$, in which, Ω is the Banach space defined by (5).

Proof. In view of Lemma 6, we can use the same steps to prove that $\mathcal{A}(\Omega)\subset\Omega$. The proof is completed.

Theorem 12
Let $\alpha,\beta,\gamma,\lambda\in\mathbb{R}$ be such that $1<\alpha\le 2$, $\lambda\in(0,|\gamma|^{-1}\Gamma(\alpha))^{\frac{1}{\alpha}}$. If

$$\frac{\lambda^{\alpha}|\beta|}{\Gamma(\alpha+1)-\alpha|\gamma|\lambda^{\alpha}}<1. \tag{14}$$

Then the problem (10)–(11) admits a unique solution on $[0,\lambda]$.

Proof. To begin the proof, we will transform the problem (10)–(11) into a fixed point problem. Define the operator $\mathcal{A}\colon \Omega \to \Omega$ by

$$\mathcal{A}f(\eta) = U + \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1}(\beta f(\xi) + \gamma\xi f'(\xi))d\xi. \tag{15}$$

Because the problem (10)–(11) is equivalent to the fractional integral equation (15), the fixed points of $\mathcal{A}$ are solutions of the problem (10)–(11).

Let $f, g \in \Omega$ be such that

$${}^C\mathcal{D}^\alpha_{0+} f(\eta) = \beta f(\eta) + \gamma\eta f'(\eta),\ {}^C\mathcal{D}^\alpha_{0+} g(\eta) = \beta g(\eta) + \gamma\eta g'(\eta).$$

This implies that

$$\mathcal{A}f(\eta) - \mathcal{A}g(\eta) = \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1}[(\beta(f(\xi) - g(\xi)) + \gamma\xi(f'(\xi) - g'(\xi)))]d\xi.$$

Also

$$|\mathcal{A}f(\eta) - \mathcal{A}g(\eta)| \le \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1}|{}^C\mathcal{D}^\alpha_{0+} f(\xi) - {}^C\mathcal{D}^\alpha_{0+} g(\xi)|d\xi. \tag{16}$$

For all $\eta \in [0, \lambda]$ we have

$$\begin{aligned} |{}^C\mathcal{D}^\alpha_{0+} f(\eta) - {}^C\mathcal{D}^\alpha_{0+} g(\eta)| &= |\beta(f(\eta) - g(\eta)) + \gamma\eta(f'(\eta) - g'(\eta))| \\ &\le |\beta||f(\eta) - g(\eta)| + \lambda|\gamma||f'(\eta) - g'(\eta)|. \end{aligned}$$

By using (4) from Remark 4 we get

$$\|{}^C\mathcal{D}^\alpha_{0+} f - {}^C\mathcal{D}^\alpha_{0+} g\|_\infty \le |\beta|\|f - g\|_\infty + \frac{|\gamma|\lambda^\alpha}{\Gamma(\alpha)}\|{}^C\mathcal{D}^\alpha_{0+} f - {}^C\mathcal{D}^\alpha_{0+} g\|_\infty.$$

As $\Gamma(\alpha) - |\gamma|\lambda^\alpha > 0$, we obtain

$$\|{}^C\mathcal{D}^\alpha_{0+} f - {}^C\mathcal{D}^\alpha_{0+} g\|_\infty \le \frac{|\beta|\Gamma(\alpha)}{\Gamma(\alpha) - |\gamma|\lambda^\alpha}\|f - g\|_\infty.$$

From (16) we find

$$\|\mathcal{A}f - \mathcal{A}g\|_\infty \le \frac{\lambda^\alpha|\beta|}{\Gamma(\alpha+1) - \alpha|\gamma|\lambda^\alpha}\|f - g\|_\infty$$

Now (14) implies that $\mathcal{A}$ is a contraction operator.

As a consequence of Theorem 7, using Banach's contraction principle [8], we deduce that $\mathcal{A}$ has a unique fixed point which is the unique solution of the problem (10)–(11) on $[0, \lambda]$. The proof is completed.

Theorem 13

Let $\lambda > 0$, $\beta, \gamma \in \mathbb{R}$, and $1 < \alpha \le 2$. If we put

$$\frac{\lambda^\alpha(|\beta| + \alpha|\gamma|)}{\Gamma(\alpha+1)} < 1. \tag{17}$$

Then the problem (10)–(11) *has at least one solution on* $[0, \lambda]$.

Proof. In the Theorem 12 we transform the problem (10)–(11) into a fixed point problem

$$\mathcal{A}f(\eta) = U + \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\xi)^{\alpha-1}(\beta f(\xi) + \gamma\xi f'(\xi))d\xi.$$

We demonstrate that $\mathcal{A}$ satisfies the assumption of Schauder's fixed point theorem 8. This could be proved through three steps

Step 1: $\mathcal{A}$ is a continuous operator.Let $(f_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim\limits_{n\to\infty} f_n = f$ in Ω. Then for each $\eta \in [0,\lambda]$,

$$|\mathcal{A}f_n(\eta) - \mathcal{A}f(\eta)| \leq \int_0^\eta \frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)} \times |\beta(f_n(\xi) - f(\xi)) + \gamma\xi(f_n'(\xi) - f'(\xi))|d\xi, \tag{18}$$

where

$${}^C\mathcal{D}^\alpha_{0^+} f_n(\eta) = \beta f_n(\eta) + \gamma\eta f_n'(\eta) \quad \text{and} \quad {}^C\mathcal{D}^\alpha_{0^+} f(\eta) = \beta f(\eta) + \gamma\eta f'(\eta).$$

We have

$$\begin{aligned}|{}^C\mathcal{D}^\alpha_{0^+} f_n(\eta) - {}^C\mathcal{D}^\alpha_{0^+} f(\eta)| &= |\beta(f_n(\eta) - f(\eta)) + \gamma\eta(f_n'(\eta) - f'(\eta))| \\ &\leq |\beta||f_n(\eta) - f(\eta)| + |\gamma|\lambda|f_n'(\eta) - f'(\eta)|.\end{aligned}$$

Inequality (4) from remark 4 yields

$$\|{}^C\mathcal{D}^\alpha_{0^+} f_n - {}^C\mathcal{D}^\alpha_{0^+} f\|_\infty \leq |\beta|\|f_n - f\|_\infty + \frac{|\gamma|\lambda^\alpha}{\Gamma(\alpha)}\|{}^C\mathcal{D}^\alpha_{0^+} f_n - {}^C\mathcal{D}^\alpha_{0^+} f\|_\infty.$$

According to (17), we have $\Gamma(\alpha) - |\gamma|\lambda^\alpha > \frac{1}{\alpha}\lambda^\alpha|\beta| > 0$, thus

$$\|{}^C\mathcal{D}^\alpha_{0^+} f_n - {}^C\mathcal{D}^\alpha_{0^+} f\|_\infty \leq \frac{|\beta|\Gamma(\alpha)}{\Gamma(\alpha) - |\gamma|\lambda^\alpha}\|f_n - f\|_\infty.$$

Since $f_n \to f$, then we get ${}^C\mathcal{D}^\alpha_{0^+} f_n \to {}^C\mathcal{D}^\alpha_{0^+} f$ as $n \to \infty$ for each $\eta \in [0,\lambda]$. Now let $K_1 > 0$ be such that for each $\eta \in [0,\lambda]$, then $|{}^C\mathcal{D}^\alpha_{0^+} f_n(\eta)| \leq K_1$ and $|{}^C\mathcal{D}^\alpha_{0^+} f(\eta)| \leq K_1$. Consequently,

$$\begin{aligned}|\mathcal{A}f_n(\eta) - \mathcal{A}f(\eta)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\xi)^{\alpha-1} \\ &\quad \times |\beta(f_n(\xi) - f(\xi)) + \gamma\xi(f_n'(\xi) - f'(\xi))|d\xi \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\xi)^{\alpha-1}|{}^C\mathcal{D}^\alpha_{0^+} f_n(\xi) -^C \mathcal{D}^\alpha_{0^+} f(\xi)|d\xi \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\xi)^{\alpha-1}[|{}^C\mathcal{D}^\alpha_{0^+} f_n(\xi)| + |{}^C\mathcal{D}^\alpha_{0^+} f(\xi)|]d\xi \\ &\leq \frac{2K_1}{\Gamma(\alpha)}\int_0^\eta (\eta-\xi)^{\alpha-1}d\xi.\end{aligned}$$

As the function $\xi \to \frac{2K_1}{\Gamma(\alpha)}(\eta-\xi)^{\alpha-1}$ is integrable on $[0,\eta]$ for every $\eta \in [0,\lambda]$, then the theorem of Lebesgue dominated convergence and (18) get us

$$|\mathcal{A}f_n(\eta) - \mathcal{A}f(\eta)| \to 0, \qquad \text{as } n \to \infty,$$

and hence

$$\lim_{n\to\infty} \|\mathcal{A}f_n - \mathcal{A}f\|_\infty = 0.$$

Consequently, $\mathcal{A}$ is continuous.

Step 2: According to (17), we set the positive real

$$r \geq \Big(1 + \frac{\lambda^\alpha|\beta|}{\Gamma(\alpha+1) - \lambda^\alpha(|\beta| + \alpha|\gamma|)}\Big)|U|,$$

and define

$$\Omega_r = \{f \in \Omega : \ \|f\|_\infty \leq r\}.$$

Observe that Ω_r is a closed, bounded and convex subset of Ω.

Let $f \in \Omega_r$ and $\mathcal{A}\colon \Omega_r \to \Omega$ be an integral operator defined by (15), then $\mathcal{A}(\Omega_r) \subset \Omega_r$. In fact, by using (4) from Lemma 4, we have for each $\eta \in [0,\lambda]$,

$$|{}^C\mathcal{D}^\alpha_{0+} f(\eta)| = |\beta f(\eta) + \gamma\eta f'(\eta)| \leq |\beta||f(\eta)| + \frac{|\gamma|\lambda^\alpha}{\Gamma(\alpha)}\|{}^C\mathcal{D}^\alpha_{0+} f\|_\infty.$$

Then

$$\|{}^C\mathcal{D}^\alpha_{0+} f\|_\infty \leq \frac{|\beta|\Gamma(\alpha)}{\Gamma(\alpha) - |\gamma|\lambda^\alpha} r$$

Hence

$$\begin{aligned}
|\mathcal{A}f(\eta)| &\leq |U| + \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\xi)^{\alpha-1}|\beta f(\xi) + \gamma\xi f'(\xi)|d\xi \\
&\leq \frac{|U|\big(1 + \frac{\lambda^\alpha|\beta|}{\Gamma(\alpha+1)-\lambda^\alpha(|\beta|+\alpha|\gamma|)}\big)}{1 + \frac{\lambda^\alpha|\beta|}{\Gamma(\alpha+1)-\lambda^\alpha(|\beta|+\alpha|\gamma|)}} + \frac{\lambda^\alpha|\beta|}{\Gamma(\alpha+1) - \alpha|\gamma|\lambda^\alpha} r \\
&\leq \frac{(\Gamma(\alpha+1) - \lambda^\alpha(|\beta| + \alpha|\gamma|))r}{\Gamma(\alpha+1) - \alpha|\gamma|\lambda^\alpha} + \frac{\lambda^\alpha|\beta|}{\Gamma(\alpha+1) - \alpha|\gamma|\lambda^\alpha} r \\
&\leq r.
\end{aligned}$$

Finally, $\mathcal{A}(\Omega_r) \subset \Omega_r$.

Step 3: $\mathcal{A}(\Omega_r)$ is relatively compact. Let $\eta_1, \eta_2 \in [0,\lambda]$, $\eta_1 < \eta_2$ and $f \in \Omega_r$. Then

$$\begin{aligned}
|\mathcal{A}f(\eta_2) - \mathcal{A}f(\eta_1)| = \Bigg|&\frac{1}{\Gamma(\alpha)}\int_0^{\eta_2} (\eta_2-\xi)^{\alpha-1}(\beta f(\xi) + \gamma\xi f'(\xi))d\xi \\
&- \frac{1}{\Gamma(\alpha)}\int_0^{\eta_1} (\eta_1-\xi)^{\alpha-1}(\beta f(\xi) + \gamma\xi f'(\xi))d\xi\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{\eta_1} |((\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1}) \\
&\quad \times (\beta f(\xi) + \gamma \xi f'(\xi))| d\xi \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\eta_1}^{\eta_2} (\eta_2-\xi)^{\alpha-1} |(\beta f(\xi) + \gamma \xi f'(\xi))| d\xi \\
&\leq \frac{|\beta| r}{\Gamma(\alpha+1) - |\gamma|\lambda^\alpha} \\
&\quad \times \left[\int_0^{\eta_1} |(\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1}| d\xi \right. \\
&\quad \left. + \int_{\eta_1}^{\eta_2} (\eta_2-\xi)^{\alpha-1} d\xi \right].
\end{aligned} \tag{19}
$$

We have

$$
(\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1} = -\frac{1}{\alpha}\frac{d}{d\xi}[(\eta_2-\xi)^\alpha - (\eta_1-\xi)^\alpha],
$$

then

$$
\int_0^{\eta_1} |(\eta_2-\xi)^{\alpha-1} - (\eta_1-\xi)^{\alpha-1}| d\xi \leq \frac{1}{\alpha}[(\eta_2-\eta_1)^\alpha + (\eta_2^\alpha - \eta_1^\alpha)].
$$

We also have

$$
\int_{\eta_1}^{\eta_2} (\eta_2-\xi)^{\alpha-1} d\xi = -\frac{1}{\alpha}[(\eta_2-\xi)^\alpha]_{\eta_1}^{\eta_2} \leq \frac{1}{\alpha}(\eta_2-\eta_1)^\alpha.
$$

Then (19) gives

$$
|\mathcal{A}f(\eta_2) - \mathcal{A}f(\eta_1)| \leq \frac{|\beta| r}{\Gamma(\alpha+1) - \alpha|\gamma|\lambda^\alpha}(2(\eta_2-\eta_1)^\alpha + (\eta_2^\alpha - \eta_1^\alpha)).
$$

As $\eta_1 \to \eta_2$, the right-hand side of the above inequality tends to zero.

As a consequence of steps 1 to 3, and by means of the Arzelà-Ascoli theorem, we deduce that $\mathcal{A}\colon \Omega_r \to \Omega_r$ is continuous, compact and also satisfies all assumptions of the fixed point theorem of Schauder's (Theorem 8). Then $\mathcal{A}$ has at least one fixed point which is a solution of the problem (10)–(11) on $[0,\lambda]$. The proof is completed.

3.3. Existence results of solutions for the original problem

In this section, we prove the existence and uniqueness of solutions of the the following problem of the space-fractional diffusion equation

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \dfrac{\partial^\alpha u}{\partial x^\alpha} + \mu u, & (x,t) \in [0,X] \times [0,\infty),\ \mu \in \mathbb{R},\ 1 < \alpha \leq 2, \\
u(0,t) = c(t)U,\ \dfrac{\partial u}{\partial x}(0,t) = 0,\ u(x,0) = f(x), & U \in \mathbb{R}.
\end{cases} \tag{20}
$$

Under the generalized self-similar form

$$u(x,t) = c(t)f(\eta) \qquad \text{with } \eta = \frac{x}{a(t)} \text{ and } a, c \in \mathbb{R}_+^*. \tag{21}$$

We denote by $(z)_+$ the positive part of z, which is z if $z > 0$ and else is zero.

Now, we give the principal theorems of this work.

Theorem 14
Let a and c be two positive real functions of t and let $\alpha, \beta, \gamma, \mu, X \in \mathbb{R}$ be such that $1 < \alpha \leq 2$. If

$$X \in (0, a^\alpha(t)|\gamma|^{-1}\Gamma(\alpha))^{\frac{1}{\alpha}} \quad \textit{and} \quad \frac{X^\alpha|\beta|}{a^\alpha(t)\Gamma(\alpha+1) - \alpha|\gamma|X^\alpha} < 1, \tag{22}$$

then for $f \in \Omega$ the problem (20) *admits a unique solution in the generalized self-similar form* (21)*, where*

$$\begin{cases} a(t) = (1-\alpha\gamma t)_+^{\frac{1}{\alpha}}, \\ c(t) = e^{\mu t}(1-\alpha\gamma t)_+^{-\frac{\beta}{\alpha\gamma}}, \end{cases} \qquad 0 < t < T \tag{23}$$

with $T > 0$ is the maximal existence time for the solution u, which may be finite or infinite. Thereupon, we separate the following cases:

(*i*) *If $\gamma < 0$, then for all $\beta, \mu \in \mathbb{R}$ the problem* (20) *admits a global solution in time under the generalized self-similar form* (21)*, this solution defined for all $t > 0$, (i.e. $T = \infty$). Moreover, if $\mu < 0$ or ($\mu = 0$ and $\beta < 0$), we have*

$$\lim_{t\to+\infty} u(x,t) = 0 \qquad \textit{for all } x \in [0, X].$$

(*ii*) *If $\gamma > 0$, the functions a and c are defined locally and well defined if and only if*

$$0 < t < T = \frac{1}{\alpha\gamma}.$$

The moment $T = \frac{1}{\alpha\gamma}$ represents the maximal existence value of the functions $a(t)$, $c(t)$.

Moreover, if $\beta > 0$, the problem (20) *admits a solution under the generalized self-similar form* (21) *which blows up in the finite time T.*

Proof. The transformation (21) reduces the space-fractional diffusion equation (20) to the ordinary differential equation of fractional order of the form

$$^C\mathcal{D}_{0^+}^\alpha f(\eta) = \beta f(\eta) + \gamma\eta f'(\eta), \qquad \beta, \gamma \in \mathbb{R}, \tag{24}$$

where $\eta = \frac{x}{a(t)}$, $a(t) > 0$ for all $t \geq 0$ and $x \in [0, X]$ for some $X \in (0, a^\alpha(t)|\gamma|^{-1}\Gamma(\alpha))^{\frac{1}{\alpha}}$ and

$$\begin{cases} \dot{c}(t) = c(t)(\beta a^{-\alpha}(t) + \mu), \\ \dot{a}(t) = -\gamma a^{1-\alpha}(t), \\ a(0) = c(0) = 1 \end{cases} \tag{25}$$

with the conditions

$$f(0) = U, \quad f'(0) = 0 \tag{26}$$

Now, to determine the functions a and c, we just solve the system (25) and after an integration from 0 to t, we get easily the system presented in (23) as follows

$$\begin{cases} a(t) = (1 - \alpha\gamma t)_+^{\frac{1}{\alpha}}, \\ c(t) = e^{\mu t}(1 - \alpha\gamma t)_+^{-\frac{\beta}{\alpha\gamma}}. \end{cases}$$

We deduce that the functions a and c are globally defined if $\gamma < 0$ and are maximal functions if $\gamma > 0$, and are well defined if and only if

$$0 < t < T = \frac{1}{\alpha\gamma}.$$

We notice from this theorem that we have two time behaviours of functions a and c, their behaviours depend on parameters of similarity β and γ.

In the case (i) (i.e. $\gamma < 0$), for every $\beta, \mu \in \mathbb{R}$ the functions a and c are defined globally in time. Moreover, we have in both cases $\mu < 0$ and ($\mu = 0$ and $\beta < 0$)

$$\lim_{t \to +\infty} u(x,t) = 0 \qquad \text{for all } x \in [0, X].$$

In the case $\gamma > 0$, we have a and c given in (23) are well defined if and only if

$$0 < t < T = \frac{1}{\alpha\gamma}.$$

We recall that the solution blows up in finite time if there exists a time $T < +\infty$ which is called the blow-up time, such that the solution is well defined for all $0 < t < T$, while

$$\sup_{x \in \mathbb{R}} |u(x,t)| \to +\infty, \qquad \text{when } t \to T^-.$$

If $\beta > 0$, the generalized self-similar solution (21) of the problem (20) is defined for all $t \in (0, T)$, the moment T represents the blow-up time of the solution such that $\lim_{t \to T^-} c(t) = +\infty$ and for all $x \in [0, X]$ we get

$$\lim_{t \to T^-} |u(x,t)| = \lim_{t \to T^-} c(t) \left| f\left(\frac{x}{a(t)}\right) \right| = +\infty \qquad \text{with } T = \frac{1}{\alpha\gamma} > 0.$$

Let $f \in \Omega$ be a continuous function. By using (21), the condition (22), is equivalent to (14), which is

$$\frac{\lambda^\alpha |\beta|}{\Gamma(\alpha + 1) - \alpha|\gamma|\lambda^\alpha} < 1. \tag{27}$$

We already proved in Theorem 12, the existence and uniqueness of a solution of the problem (24)–(26) provided that (27) holds true. Consequently, if (22) holds for any $(x,t) \in [0, X] \times [0, \infty)$, then there exists a unique solution of the problem of the space-fractional diffusion equation (20) under the generalized self-similar form (21). The proof is completed.

Theorem 15
Let $\alpha, \beta, \gamma, \mu, X \in \mathbb{R}$, be such that $1 < \alpha \leq 2$ and $X > 0$. If

$$\frac{X^{\alpha}(|\beta| + \alpha|\gamma|)}{a^{\alpha}(t)\Gamma(\alpha+1)} < 1. \tag{28}$$

Then, for $f \in \Omega_r$, the problem (20) *has at least one solution in the generalized self-similar form* (21) *which is global in time when $\gamma < 0$ and blows up in a finite time $0 < t < T = \frac{1}{\alpha\gamma}$, when $\beta, \gamma > 0$.*

Proof. Based on Theorem 13, we use the same steps through which we proved Theorem 14 to prove the global existence and blow-up of a generalized self-similar solution to the problem (20) provided that the condition (28) holds true. The proof is completed.

4. Conclusion

This paper discussed the existence and uniqueness of solutions for a class of space-fractional diffusion equations with mixed conditions under the generalized self-similar form. The behavior of these solutions depends on parameters that satisfy certain conditions, and which make their existence global or local in finite time T. For that matter, we used the Banach contraction principle and Schauder's fixed point theorem, while the differential operator used is the Caputo fractional derivative.

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