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Examples of non connective $C^\star$-algebras

Abstract. This paper investigates the problem of the existence and uniqueness of solutions under the generalized self-similar forms to the space-fractional diffusion equation. Therefore, through applying the properties of Schauder's and Banach's fixed point theorems; we establish several results on the global existence and blow-up of generalized self-similar solutions to this equation.

1. Introduction

For a Hilbert space $\mathcal{H}$, we denote by $L(\mathcal{H})$ the $C^\star$-algebra of bounded and linear operators on $\mathcal{H}$. The ideal of compact operators is denoted by $\mathcal{K} \subset L(\mathcal{H})$. For the $C^\star$-algebra A, the cone over A is defined as $CA = C_0[0,1) \otimes A$, the suspension of A as $SA = C_0(0,1) \otimes A$.

Definition 1
Let A be a $C^\star$-algebra and $n \in \mathbb{N}, n \geq 1$. A is *connective* if there is a $\star$-monomorphism

$$\Phi \colon A \to \prod_n CL(\mathcal{H}) / \bigoplus_n CL(\mathcal{H}),$$

which is liftable to a completely positive and contractive map $\phi \colon A \to \prod_n CL(\mathcal{H})$.

For a discrete group G, we define $I(G)$ to be the augmention ideal, i.e. the kernel of the trivial representation $C^\star(G) \to \mathbb{C}$. Group G is called connective if $I(G)$

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is a connective $C^\star$-algebra. From definition (see [4, p. 4921]) connectivity of G may be viewed as a stringent topological property that accounts simultaneously for the quasidiagonality of $C^\star(G)$ and the verification of the Kadison-Kaplansky conjecture for certain classes of groups. Here we can referring to conjecture from 2014 [2, p. 166]: *If G is a discrete, countable, torsion-free, amenable group, then the natural map*

$$[[I(G), \mathcal{K}]] \to KK(I(G), \mathcal{K}) \cong K^0(I(G))$$

is an isomorphism of groups. Here $KK(I(G), \mathcal{K})$ is the Kasparov group and $[[I(G), \mathcal{K}]]$ is a group of the homotopy classes of asymptotic morphisms. In 2017 M. Dadarlat found an amenable and not connective group G_2 for which the above conjecture fails [6, Cor. 3.2].

Connective groups must be torsion-free, [3, Remark 2.8 and 4.4]. Here is a short list of such groups:

1. a countable torsion free nilpotent groups, [3, Th.4.3];
2. let $0 \to N \to G \to H \to 0$ be a central extension of discrete countable amenable groups where N is torsion-free. If H is connective then so does G; [3, Th. 4.1];
3. wreath product of connective groups is a connective group [5, Th.3.2];
4. a torsion-free crystallographic group is connective if and only if is locally indicable if and only if is diffuse (see below) and [6].

A discrete group G is called *locally indicable* if every finitely generated non-trivial subgroup L of G has an infinite abelianization. The group G is called *diffuse* if every non-empty finite subset A of G has an element $a \in A$ such that for any $g \in G$, either ga or $g^{-1}a$ is not in A, see [6] and [7]. More examples of nonabelian connective groups were exhibited in [4], [5] and[6].

Dadarlat's group G_2 is a 3-dimensional, torsion-free crystallographic group, where a crystallographic group Γ, of dimension n is a cocompact and discrete subgroup of the isometry group $E(n) = O(n) \ltimes \mathbb{R}^n$ of the Euclidean space $\mathbb{R}^n$. Group Γ is cocompact if and only if the orbit space $E(n)/\Gamma$ is compact.

From Bieberbach theorems (see [9, Chapter 1]) any crystallographic group Γ defines a short exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \to H \to 0,$$

where a free abelian group $\mathbb{Z}^n$ is a maximal abelian subgroup and H is a finite group. Group H is sometimes called a holonomy group of Γ. The above group G_2 is isomorphic to the subgroup $E(3)$ and is generated by

$$G_2 \cong \text{gen}\left\{ A = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \left(\frac{1}{2}, \frac{1}{2}, 0\right) \right), B = \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \left(0, \frac{1}{2}, \frac{1}{2}\right) \right) \right\}.$$

A torsion-free crystallographic group is called a Bieberbach group. The orbit space $\mathbb{R}^n/\Gamma$ of a Bieberbach group is a n-dimensional closed flat Riemannian manifold M with holonomy group isomorphic to H.

A general characterization of connective Bieberbach groups is given in [6]. The following two theorems give us a landscape of them.

Theorem 1 ([6, Theorem 1.2])
Let Γ be a Bieberbach group. The following assertions are equivalent.

1. Γ *is connective*
2. *Every nontrivial subgroup of* Γ *has a nontrivial center.*
3. Γ *is a poly-*$\mathbb{Z}$ *group*
4. $\widehat{G} \setminus \{\iota\}$ *has no nonempty compact open subsets.*

The unitary dual $\widehat{G}$ *of* G *consists of equivalence classes of irreducible unitary representations of* G *and* ι *denotes the trivial representation.*

Theorem 2 ([6, Theorem 1.1])
A Bieberbach group with a finite abelianization is not connective.

In our note we give an example of two infinite families of not connective groups. Both of them are generalization of the 3-dimensional Hantzsche-Wendt group G_2.

2. Examples

Example 1 ([9, Definition 9.1])
Let $n \geq 3$. By generalized Hantzsche-Wendt (GHW for short) group we shall understand any torison-free crystallographic groups of rank n with a holonomy group $(\mathbb{Z}_2)^{n-1}$.

Example 2 ([1, Definition], [10, Definition 1])
Let $n \geq 0$. A group

$$G_n = \{x_1, x_2, \ldots, x_n : \ x_i^{-1} x_j^2 x_i x_j^2 \quad \text{for all } i \neq j\}$$

we shall call a combinatorial Hantzsche-Wendt group.

For the properties of GHW groups we refer to [9, Chapter 9]. We have $G_0 = 1$ and $G_1 \simeq \mathbb{Z}$. Combinatorial Hantzsche-Wendt groups are torsion-free, see [1, Theorem 3.3] and for $n \geq 2$ are nonunique product groups. A group G is called a unique product group if given two nonempty finite subset X, Y of G, there exists at least one element $g \in G$ which has a unique representation $g = xy$ with $x \in X$ and $y \in Y$. We are ready to present our main result.

Proposition 1
Generalized Hantzsche-Wendt groups with trivial center and nonabelian, combinatorial Hantzsche-Wendt groups are not connective.

Proof. From [3, Remark 2.8 (i)] the connectivity property is inherited by subgroups. Let G be any group from family of GHW groups or family of combinatorial Hanzsche-Wendt groups. In both cases a group G_2 is a subgroup of G. In the first case it follows from [9, Proposition 9.7]. In the second case it follows from definition, see [1, Prop. 3.4].

Note that in the case of GHW groups we can also use Theorem 2, since all these groups have a finite abelianizations.

Remark 1
From [10], for $n \geq 3$, G_n has a non-abelian free subgroup. Hence is not amenable.

Remark 2
The counterexample to the Kaplansky unit conjecture was given in 2021 by G. Gardam [8]. It was found in the group ring $\mathbb{F}_2[G_2]$. The Kaplansky unit conjecture states that every unit in $K[G]$ is of the form kg for $k \in K \setminus \{0\}$ and $g \in G$.

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