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Equivalential algebras with conjunction on the regular elements

Abstract. We introduce the definition of the three-element equivalential algebra $\mathbf{R}$ with conjunction on the regular elements. We study the variety generated by $\mathbf{R}$ and prove the Representation Theorem. Then, we construct the finitely generated free algebras and compute the free spectra in this variety.

1. Introduction

In this paper, we investigate the equivalential algebra with conjunction on the regular elements, called $\mathbf{R}$. It is a reduct of type $(2, 2, 0)$ of the three-element Heyting algebra in which are the distinguished constant 1, a binary operation corresponding to the equivalential term $\leftrightarrow$ and an additional binary operation, which is conjunction on the regular elements.

It is known that if V is a Fregean variety then V is of type 2 or 3 [7, p. 606] in the sense of Tame Congruence Theory of Hobby and McKenzie [4]. Every equivalential algebra is solvable, so it is of type 2 [7, p. 606]. However, every Heyting algebra is congruence distributive and it is of type 3. The purpose of the paper was to investigate of the smallest fregean algebra of mixed type. An example of such an algebra is $\mathbf{R}$.

In this case, the second binary operation (besides $\leftrightarrow$), is not conjunction on all elements (in contrast to, for example, the Skolem algebra), but is conjunction only on the regular elements. These elements play an important role in Heyting algebras, where an element a is said to be regular if $\neg\neg a = a$, because they are essential in the study of the relationships between classical and intuitionistic logic.

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The variety generated by $\mathbf{R}$, denoted by $\mathcal{V}(\mathbf{R})$, is locally finite, so we can study the free spectra. For this purpose, we introduce representation of finite algebras in $\mathcal{V}(\mathbf{R})$.

The paper is organized as follows. In Section 2 we give the most important definitions and theorems, which are used throughout the paper, related to the notions of the Fregean variety and the equivalential algebra. Next, we introduce the definition of the three-element equivalential algebra with conjunction on the regular elements. In Section 3, we describe the frame for any algebra in $\mathcal{V}(\mathbf{R})$ and we prove the representation theorem for finite algebras from $\mathcal{V}(\mathbf{R})$. Finally, we construct the free algebra in $\mathcal{V}(\mathbf{R})$ with a finite number of generators and we find the formula for the free spectrum in this variety.

2. Preliminaries

We start with the notion of a Fregean algebra.

Definition 1 ([7, p. 597])
An algebra $\mathbf{A}$ with a distinguished constant 1 is called *Fregean* if $\mathbf{A}$ is:

1. 1-*regular*, i.e. $1/\alpha = 1/\beta$ implies $\alpha = \beta$ for all $\alpha, \beta \in \operatorname{Con} \mathbf{A}$;
2. *congruence orderable*, i.e. $\Theta_{\mathbf{A}}(1, a) = \Theta_{\mathbf{A}}(1, b)$ implies $a = b$ for all $a, b \in A$.

A variety $\mathcal{V}$ is said to be Fregean if all its algebras are Fregean. Let $\mathbf{A} \in \mathcal{V}$. Congruence orderability allows to introduce a natural partial order on the universe of $\mathbf{A}$: $a \leq b \Leftrightarrow_{df} \Theta_{\mathbf{A}}(1, b) \subseteq \Theta_{\mathbf{A}}(1, a)$. Moreover, the Fregean varieties are congruence modular (see [3]). A natural example of a Fregean variety is a variety of equivalential algebras.

Definition 2
An equivalential algebra is an algebra $(A, \leftrightarrow, 1)$ of type $(2, 0)$ that is a subreduct of a Heyting algebra with the binary operation $\leftrightarrow$ given by

$$x \leftrightarrow y := (x \to y) \wedge (y \to x).$$

In 1975 J. K. Kabziński and A. Wroński proved that the class $\mathbf{E}$ of all equivalential algebras is equationally definable by identities, so it forms a variety, [8]. Similar to literature, we adopt the convention of associating to the left and ignoring (or replacing with $\cdot$) the symbol of equivalence operation.

The class $\mathbf{E}$ is congruence permutable (it follows from [7, p. 598]). Moreover, equivalential algebras form a paradigm for congruence permutable Fregean varieties, as the following theorem shows:

Theorem 3 ([7, Theorem 3.8])
Let $\mathcal{V}$ be a congruence permutable Fregean variety. Then there exists a binary term $\leftrightarrow$ such that for every $\mathbf{A} \in \mathcal{V}$:

1. $(A, \leftrightarrow, 1)$ *is an equivalential algebra;*
2. $\leftrightarrow$ *is a principal congruence term of* $\mathbf{A}$*, i.e.* $(a, b) \in \alpha$ *iff* $(1, a \leftrightarrow b) \in \alpha$ *for every* $\alpha \in \operatorname{Con} \mathbf{A}$.

If $\mathcal{V}$ is a congruence permutable Fregean variety and $\mathbf{A} \in \mathcal{V}$, then we will denote an equivalential reduct of $\mathbf{A}$ by $\mathbf{A}^e$. In a Fregean variety the subdirectly irreducible algebras can be characterized as those which have the largest non-unit element, which will be denoted by $*$. If A is a subdirectly irreducible algebra with the monolith μ then by [7, Lemma 2.1] we have that $1/\mu = \{1, *\}$ and all other μ-cosets are singletons.

2.1. Definition and basic properties

Definition 4
An equivalential algebra with conjunction on the regular elements is an algebra $\mathbf{R} := (\{0, *, 1\}, \leftrightarrow, r, 1)$ of type $(2, 2, 0)$, where $(\{0, *, 1\}, \leftrightarrow, 1)$ is an equivalential algebra and r is a binary commutative operation presented in the table below (on the right):

$\leftrightarrow$	1	$*$	0
1	1	$*$	0
$*$	$*$	1	0
0	0	0	1

r	1	$*$	0
1	1	1	0
$*$	1	1	0
0	0	0	0

Then $\mathbf{R} = (\{0, *, 1\}, \leftrightarrow, r, 1)$ is a reduct of the three-element Heyting algebra with an order: $0 < * < 1$, in which $x \leftrightarrow y = (x \to y) \wedge (y \to x)$ and $r(x, y) = [(x \leftrightarrow 0) \leftrightarrow 0] \wedge [(y \leftrightarrow 0) \leftrightarrow 0]$ (briefly: $r(x, y) = x00 \wedge y00$). Note that we can write the operation r without $\leftrightarrow$, because the following identity is true in Heyting algebras: $(x \leftrightarrow 0) \leftrightarrow 0 = (x \to 0) \to 0$.

The variety generated by $\mathbf{R}$ will be denoted by $\mathcal{V}(\mathbf{R})$. It is easy to show that $\mathbf{R}$ is a Fregean algebra.

Remark 5
$\mathbf{R}$ has two nontrivial subalgebras:
$\mathbf{2} := (\{*, 1\}, \leftrightarrow, r, 1)$, where $r \equiv 1$;
$\mathbf{2}^\wedge := (\{0, 1\}, \leftrightarrow, r, 1)$, where $r(x, y) = x \wedge y$.

Remark 6
$\operatorname{Con} \mathbf{R} = \{\mathbf{1_R}, \mu_\mathbf{R}, \mathbf{0_R}\}$ is a three-element chain (under $\subseteq$), so $\mathbf{R}$ is subdirectly irreducible, where $\mu_\mathbf{R}$ is the monolith of $\mathbf{R}$. Furthermore, $\mathbf{R}/\mu_\mathbf{R} \cong \mathbf{2}^\wedge$.

Proposition 7
The variety $\mathcal{V}(\mathbf{R})$ is a Fregean variety.

Proof. According to the above remark, if $\mathbf{A}$ is not trivial, then $\mathbf{A} \in H(\mathbf{R})$ iff $\mathbf{A} \cong \mathbf{2}^\wedge$ or $\mathbf{A} \cong \mathbf{R}$. Thus $\mathbf{A} \in HS(\mathbf{R})$ iff $A \cong \mathbf{R}$ or $\mathbf{A} \cong \mathbf{2}$ or $\mathbf{A} \cong \mathbf{2}^\wedge$. Therefore all algebras in $HS(\mathbf{R})$ are congruence orderable. Furthermore, since all equivalential algebras are congruence $\mathbf{1}$-regular and this property is preserved under expansions of a language, so $\mathcal{V}(\mathbf{R})$ is $\mathbf{1}$-regular too. Consequently, we conclude from Theorem 2.10 [7, p. 603] that $\mathcal{V}(\mathbf{R})$ is a Fregean variety.

Using Theorem 10.12 [2, p. 97] one can deduce the following result:

Proposition 8
There are only three (up to isomorphism) nontrivial subdirectly irreducible algebras in $\mathcal{V}(\mathbf{R})$: $\mathbf{R}$, $\mathbf{2}$ and $\mathbf{2}^{\wedge}$.

3. Representation

Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$. We say that $\mu \in \mathrm{Con}\,\mathbf{A}$ is *completely meet-irreducible* if $\mu \neq A^2$ and for any family $\{\mu_i : i \in I\} \subseteq \mathrm{Con}\,\mathbf{A}$ such that $\mu = \bigcap_{i \in I} \mu_i$, we have $\mu = \mu_i$ for some $i \in I$. If μ is completely meet-irreducible, then there exists the unique cover of μ in $\mathrm{Con}\,\mathbf{A}$, denoted by μ^+, i.e. $\mu^+ = \bigcap\{\gamma \in \mathrm{Con}\,\mathbf{A} : \mu < \gamma\}$. The set of all completely meet-irreducible congruences will be denoted by $\mathrm{Cm}(\mathbf{A})$. Similarly, we can define a *completely join-irreducible* congruence ν and in this situation ν covers the unique element, denoted by ν^-.

It is folklore that μ is completely meet-irreducible in $\mathrm{Con}\,\mathbf{A}$ iff $\mathbf{A}/\mu$ is subdirectly irreducible. Thus, we conclude from Proposition 8 that $\mu \in \mathrm{Cm}(\mathbf{A})$ iff $\mathbf{A}/\mu \cong \mathbf{k}$ for $\mathbf{k} \in \{\mathbf{R}, \mathbf{2}, \mathbf{2}^{\wedge}\}$.

We use the following notation:

$$\begin{aligned}
\widetilde{L} &:= \{\mu \in \mathrm{Cm}(\mathbf{A}) : \ \mathbf{A}/\mu \cong \mathbf{2}^{\wedge}\},\\
\underline{L} &:= \{\mu \in \mathrm{Cm}(\mathbf{A}) : \ \mathbf{A}/\mu \cong \mathbf{R}\},\\
P &:= \{\mu \in \mathrm{Cm}(\mathbf{A}) : \ \mathbf{A}/\mu \cong \mathbf{2}\},\\
L &:= \widetilde{L} \cup \underline{L}.
\end{aligned}$$

Definition 9
Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ and $\varphi, \psi \in \mathrm{Cm}(\mathbf{A})$. We introduce a binary relation $\sim$ on $\mathrm{Cm}(\mathbf{A})$ as follows:

$$\varphi \sim \psi \overset{def}{\Leftrightarrow} \varphi = \psi \text{ or } \varphi, \psi \in P \text{ or } (\varphi, \psi \in \underline{L} \text{ and } \varphi^+ = \psi^+).$$

It is easy to see that $\sim$ is the equivalence relation. Note that in the case of algebras in $\mathcal{V}(\mathbf{R})$, the relation $\sim$ is the same as the relation defined in [5, p. 51], which is related to the concept of projectivity.

If $\mathbf{A} \in \mathcal{V}(\mathbf{R})$, $\mu \in \mathrm{Cm}(\mathbf{A})$ and $U \subseteq \mu/_{\sim}$, we will denote $\overline{U} := U \cup \{\mu^+\}$ and $0_U := \bigcap U$.

Definition 10
Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ and let $\leq$ be the order on $\mathrm{Con}\,\mathbf{A}$ restricted to $\mathrm{Cm}(\mathbf{A})$, i.e.

$$\varphi \leq \psi \Leftrightarrow \varphi \subseteq \psi.$$

The structure $\mathbf{Cm}(\mathbf{A}) := (\mathrm{Cm}(\mathbf{A}), \leq, \sim)$ is called a frame of $\mathbf{A}$.

Lemma 11
Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ and let $\mu \in \mathrm{Cm}(\mathbf{A})$ be such that $\mathbf{A}/\mu \cong \mathbf{R}$. Then $\mathbf{A}/\mu^+ \cong \mathbf{2}^{\wedge}$.

Proof. Let $f\colon \mathbf{A}/\mu \to \mathbf{2}^\wedge$ be a function such that $f(1/\mu) = f(*/\mu) = 1$ and $f(0/\mu) = 0$. Then f is a surjective homomorphism and $\ker f = \mu^+/\mu$. So $\mathbf{A}/\mu/_{\mu^+/\mu} \cong \mathbf{2}^\wedge$. Therefore $\mathbf{A}/\mu^+ \cong \mathbf{2}^\wedge$.

Corollary 12
Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ and $\varphi, \psi \in \mathrm{Cm}(\mathbf{A})$. Then $\varphi \leq \psi$ iff $\varphi = \psi$ or ($\varphi \in \underline{L}$, $\psi \in \widetilde{L}$ and $\varphi^+ = \psi$).

Since the Fregean varieties are congruence modular, so we have the following Lemma.

Lemma 13 ([5, p. 51, Lemma 22])
Let $\mathbf{A} \in \mathcal{V}$, where $\mathcal{V}$ is a Fregean variety. Then for all $\alpha, \beta \in \mathrm{Con}\,\mathbf{A}$ and $\mu \in \mathrm{Cm}(\mathbf{A})$ with $\alpha \wedge \beta \leq \mu$ there are $\mu_1, \mu_2 \in \mu/_\sim \cup \{1_\mathbf{A}\}$ such that $\alpha \leq \mu_1, \beta \leq \mu_2$ and $\mu_1 \wedge \mu_2 \leq \mu$.

Proposition 14
Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ be finite. If $W \subseteq L$ and $\mu \in P$, then $\bigcap W \not\leq \mu$. Similarly, if $W \subseteq P$ and $\mu \in L$, then $\bigcap W \not\leq \mu$.

Proof. Suppose, contrary to our claim, that $W = \{\varphi_1, \dots, \varphi_n\} \subseteq L$ and $\varphi_1 \wedge \cdots \wedge \varphi_n \leq \mu$ for some $\mu \in P$. From Lemma 13 there exist $\mu_1, \mu_2 \in \mu/_\sim \cup \{1_\mathbf{A}\}$, such that $\varphi_1 \leq \mu_1, \varphi_2 \wedge \cdots \wedge \varphi_n \leq \mu_2$ and $\mu_1 \wedge \mu_2 \leq \mu$. By Corollary 12, each element of $\mu/_\sim$ is incomparable with φ_1, so $\mu_1 = 1_\mathbf{A}$. Thus $\varphi_2 \wedge \cdots \wedge \varphi_n = \varphi_1 \wedge \cdots \wedge \varphi_n \leq \mu$. Doing likewise, after $(n-1)$ steps, we get that $\varphi_n \leq \mu$, a contradiction. Similar considerations apply to the second sentence.

The symmetric difference of X and Y will be denoted by $X \div Y$ and the complement of a set X will be denoted by X'.

Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ and $\mu \in \mathrm{Cm}(\mathbf{A})$. We introduce a binary operation $\bullet$ on a set $\overline{\mu/_\sim}$ as follows

$$\mu_1 \bullet \mu_2 := (\mu_1 \div \mu_2)' \cap \mu^+$$

for $\mu_1, \mu_2 \in \overline{\mu/_\sim}$. Note that $\mu_1 \bullet \mu_2 = \mu^+$ for $\mu_1 = \mu_2$ and recall that if $\mu_1, \mu_2 \in \mu/_\sim$, then $\mu_1^+ = \mu_2^+ = \mu^+$. Clearly, the operation $\bullet$ on the subsets μ^+ forms a Boolean group. We show that $(\overline{\mu/_\sim}, \bullet)$ is its subgroup.

Theorem 15
Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ and $\mu \in \mathrm{Cm}(\mathbf{A})$. Then $(\overline{\mu/_\sim}, \bullet)$ forms a Boolean group.

Proof. Let $\mu_1, \mu_2 \in \mu/_\sim$ and $\mu_1 \neq \mu_2$. We first prove that $\mu_1 \bullet \mu_2$ is a congruence on $\mathbf{A}$. From [6, Lemma 4.1] $\mu_1, \mu_2 \in \mathrm{Cm}(\mathbf{A}^e)$. Thus, from the proof of Proposition 3 [10] we obtain $\mu_1 \bullet \mu_2 \in \mathrm{Con}\,\mathbf{A}^e$. We show that relation $\mu_1 \bullet \mu_2$ is compatible with the operation r. Choose $(a,b), (c,d) \in \mu_1 \bullet \mu_2$. We need to consider the following three cases:

1) $\mu \in \underline{L}$. We want to show that $(r(a,c), r(b,d)) \in \mu_1 \bullet \mu_2$. On the contrary, suppose that $(r(a,c), r(b,d)) \notin \mu_1 \bullet \mu_2$. Thus, without loss of generality, we can assume that $(r(a,c), r(b,d)) \in \mu_1 \setminus \mu_2$, so $r(a,c) \cdot r(b,d) \in 1/\mu_1$ and $r(a,c) \cdot r(b,d) \notin 1/\mu_2$. Since $(a,b) \in \mu_1 \subseteq \mu_2^+$, $(c,d) \in \mu_2 \subseteq \mu_2^+$

and in addition μ_2^+ is a congruence, we have that $(r(a,c), r(b,d)) \in \mu_2^+$ and consequently

$$r(a,c) \cdot r(b,d) \in (1/\mu_2^+) \setminus (1/\mu_2) = \{*/\mu_2\}.$$

Therefore $r(a,c) \cdot r(b,d)/\mu_2 = */\mu_2$, thus $r(a,c)/\mu_2 \cdot r(b,d)/\mu_2 = */\mu_2$. It follows that $r(a,c)/\mu_2 = 1/\mu_2$ and $r(b,d)/\mu_2 = */\mu_2$ (or vice versa). Thus $r^{\mathbf{A}/\mu_2}(b/\mu_2, d/\mu_2) = */\mu_2$, contrary to the definition of r. Hence $\mu_1 \bullet \mu_2$ is a congruence on $\mathbf{A}$.

2) $\mu \in \widetilde{L}$. Then $\mu_1 = \mu_2$, so $\mu_1 \bullet \mu_2 = 1_{\mathbf{A}}$ and consequently $(r(a,c), r(b,d)) \in \mu_1 \bullet \mu_2$.

3) $\mu \in P$. Since r is constantly equal to 1 on $\mathbf{A}/\mu_1$ and $\mathbf{A}/\mu_2$, thus

$$r(a,c) \cdot r(b,d) \in 1/\mu_1 \quad \text{and} \quad r(a,c) \cdot r(b,d) \in 1/\mu_2.$$

Therefore $r(a,c) \cdot r(b,d) \in 1/\mu_1 \wedge \mu_2$ and so $(r(a,c), r(b,d)) \in \mu_1 \wedge \mu_2 \subseteq \mu_1 \bullet \mu_2$. Consequently, $\mu_1 \bullet \mu_2$ is a congruence on $\mathbf{A}$.

Our next claim is that $\mu_1 \bullet \mu_2 \in \mathrm{Cm}(\mathbf{A})$. From Birkhoff's theorem [9, p. 49] there exists $\{\varphi_i\}_{i\in I} \subseteq \mathrm{Cm}(\mathbf{A})$ such that $\mu_1 \bullet \mu_2 = \bigwedge_{i\in I} \varphi_i$. Since from [6, Lemma 4.1] $\mathrm{Cm}(\mathbf{A}) \subseteq \mathrm{Cm}(\mathbf{A}^e)$ and $\mu_1 \bullet \mu_2 \in \mathrm{Cm}(\mathbf{A}^e)$, thus there exists $i \in I$ such that $\mu_1 \bullet \mu_2 = \varphi_i$.

It remains to prove that $\mu_1 \bullet \mu_2 \sim \mu$. Note that $(\mu_1 \bullet \mu_2)^+ = \mu_1^+ = \mu^+$. Let us consider the cases:

1) $\mu \in \underline{L}$. From equality $(\mu_1 \bullet \mu_2)^+ = \mu^+$ we get immediately $\mu_1 \bullet \mu_2 \sim \mu$.

2) $\mu \in P$. Then $\mu_1 \bullet \mu_2 \in \widetilde{L} \cup P$. Suppose that $\mu_1 \bullet \mu_2 \in \widetilde{L}$. Thus $\mu_1 \wedge \mu_2 \leq \mu_1 \bullet \mu_2$ contrary to Proposition 14. Thus $\mu_1 \bullet \mu_2 \in P$ and so $\mu_1 \bullet \mu_2 \sim \mu$.

A maximal proper subalgebra of the Boolean group is called a *hyperplane*. For $Z \subseteq \mathrm{Cm}(\mathbf{A})$, we will write $Z{\uparrow} := \{\nu \in \mathrm{Cm}(\mathbf{A}) : \ \nu \geq \mu \text{ for some } \mu \in Z\}$ and analogously $Z{\downarrow} = \{\nu \in \mathrm{Cm}(\mathbf{A}) : \ \nu \leq \mu \text{ for some } \mu \in Z\}$.

Now, following Słomczyńska [10], we introduce the notion of the hereditary set.

Definition 16

Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ and $Z \subseteq \mathrm{Cm}(\mathbf{A})$. We say that Z is *hereditary* if:

1. $Z = Z{\uparrow}$;
2. $(\overline{Z \cap P}, \bullet)$ is a hyperplane in $(\overline{P}, \bullet)$ or $P \subseteq Z$;
3. for all $\mu \in \underline{L}$, if $\mu^+ \in Z$, then $\mu/_\sim \subseteq Z$ or $(\overline{\mu/_\sim \cap Z}, \bullet)$ is a hyperplane in $(\overline{\mu/_\sim}, \bullet)$.

We denote the set of all hereditary subsets of $\mathrm{Cm}(\mathbf{A})$ by $\mathcal{H}(\mathbf{A})$. Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$. We say that $a \in A \setminus \{1\}$ is *irreducible* if $\Theta(1,a)$ is completely join-irreducible in $\mathrm{Con}\,\mathbf{A}$. We denote the set of all irreducible elements in $\mathbf{A}$ by $I(\mathbf{A})$.

Next, we define a map M in the following way:

$$M\colon A \ni a \to M(a) := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \ a \in 1/\mu\}$$

for all $\mathbf{A} \in \mathcal{V}(\mathbf{R})$.

Lemma 17

Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ be finite. Let $\mu \in \mathrm{Cm}(\mathbf{A})$, $U = \mu/_\sim$ and let $(\overline{H}, \bullet)$ be a hyperplane in $(\overline{U}, \bullet)$. Then there exists $a \in I(A)$ such that $(1, a) \in (\mu^+ \setminus 0_U)$ and $\overline{H} = \overline{M(a) \cap U}$.

Proof. We first prove that if $\nu, \phi, \psi \in U$ are different in pairs and $\phi \wedge \psi \leq \nu$, then $\nu \in \{\phi, \psi, \phi \bullet \psi\}$. Suppose the assertion is false. Thus there exists $(1, a) \in \phi \setminus \nu$, $(1, b) \in \psi \setminus \nu$ and $(1, c) \in (\phi \bullet \psi) \setminus \nu$. Let $d := a \cdot b \cdot c$. Then $(1, d) \in \nu^+$ and $d/\nu = */\nu$. Consequently, $(1, d) \notin \nu$.

On the other hand, $\phi \wedge \psi = \phi \wedge (\phi \bullet \psi) = \psi \wedge (\phi \bullet \psi) \leq \nu$. Thus $(1, a) \in \psi' \wedge (\phi \bullet \psi)'$, $(1, b) \in \phi' \wedge (\phi \bullet \psi)'$, $(1, c) \in \phi' \wedge \psi'$. Then $d/\phi = 1/\phi$ and $d/\psi = 1/\psi$. Therefore $(1, d) \in \phi \wedge \psi \leq \nu$, a contradiction. Subsequently, if we use Lemma 13 and apply induction, we can show that if $\mu_0, \mu_1, \ldots, \mu_n \in U$ and $\mu_1 \wedge \cdots \wedge \mu_n \leq \mu_0$, then there exist $i_1, \ldots, i_k \in \{1, \ldots, n\}$ such that $\mu_{i_1} \bullet \cdots \bullet \mu_{i_k} = \mu_0$.

Note that, if $\bigwedge H = 0_U$, then for all $\nu \in U$ there would exist $\mu_1, \ldots, \mu_k \in H$ such that $\mu_1 \bullet \cdots \bullet \mu_k = \nu$, contrary to the assumption that $(\overline{H}, \bullet)$ is a subalgebra of $(\overline{U}, \bullet)$. Thus $0_U \subsetneq \bigwedge H$.

Thus there exists $a \in I(A)$ such that $\Theta(1, a)$ is completely join-irreducible, $\Theta(1, a) \subseteq \bigwedge H$, but $\Theta(1, a) \nsubseteq 0_U$. Therefore $(1, a) \in \bigwedge H$ and $(1, a) \in (\mu^+ \setminus 0_U)$. Thus $H \subseteq M(a)$ and there exists $\nu \in U$ such that $\nu \notin M(a)$. Hence $\overline{H} \subseteq \overline{M(a) \cap U} \subsetneq \overline{U}$ and consequently from maximality of $\overline{H}$ we get that $\overline{H} = \overline{M(a) \cap U}$.

Lemma 18

Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ be finite and $a \in I(\mathbf{A})$. Then there exists a unique $U = \mu/_\sim \in \mathrm{Cm}(\mathbf{A})/_\sim$ such that $(1, a) \in \mu^+$ and $(\overline{M(a) \cap U}, \bullet)$ is a hyperplane in $(\overline{U}, \bullet)$.

Proof. Since $a \neq 1$, it follows that $M(a) \neq \mathrm{Cm}(\mathbf{A})$, and so there exists $\gamma \in \mathrm{Cm}(\mathbf{A})$ such that $\gamma \notin M(a)$. Then $(\overline{\gamma/_\sim \cap M(a)}, \bullet)$ is a hyperplane in $(\overline{\gamma/_\sim}, \bullet)$ because it is a subgroup of $(\overline{\gamma/_\sim}, \bullet)$ and, as is easy to check, $(\gamma/_\sim \cap M(a)) \cup \{\varphi\}$ generates $\overline{\gamma/_\sim}$ for all $\varphi \notin \gamma/_\sim \cap M(a)$.

To prove a uniqueness, suppose, contrary to our claim, that there exist $U, W \in \mathrm{Cm}(\mathbf{A})/_\sim$, $U \neq W$, which meet the assumptions. Then there exist $\mu, \nu \in \mathrm{Cm}(\mathbf{A})$ such that $\mu \in U \setminus M(a)$ and $\nu \in W \setminus M(a)$. Since $a \notin 1/\mu$ and $a \in 1/\mu^+$ so $\Theta(1, a) \vee \mu = \mu^+$ and $\Theta(1, a) \wedge \mu \leq \Theta^-(1, a)$. Hence $\Theta(1, a) \wedge \mu = \Theta^-(1, a)$, since otherwise we would get a contradiction with modularity of $\mathrm{Con}\,\mathbf{A}$. By a similar argument we get $\Theta(1, a) \vee \nu = \nu^+$ and $\Theta(1, a) \wedge \nu = \Theta^-(1, a)$. Consequently, $\mu \sim \nu$, contrary to $U \neq W$.

Also note that if $\mathbf{A} \in \mathcal{V}(\mathbf{R})$, $\mu \in \mathrm{Cm}(\mathbf{A})$ and $(1, a) \in \mu^+$ so $(\overline{M(a) \cap \mu/_\sim}, \bullet)$ is a hyperplane in $(\overline{\mu/_\sim}, \bullet)$ or $\mu/_\sim \subseteq M(a)$. Therefore, the following conclusion follows.

Corollary 19

Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ be finite and $a \in A$. Then $M(a)$ is a hereditary set.

The following theorem is the key to construct the finitely generated free algebras.

Theorem 20
Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ *and* $\mathbf{A}$ *be finite. Then the map* $M\colon A \ni a \to M(a) := \{\mu \in \mathrm{Cm}(\mathbf{A}) : a \in 1/\mu\}$ *establishes the isomorphism between* $\mathbf{A}$ *and* $(\mathcal{H}(\mathbf{A}), \leftrightarrow, \mathit{r}, \mathbf{\mathit{1}})$, *where*

$$\begin{aligned} Z \leftrightarrow Y &:= ((Z \div Y){\downarrow})', \\ \mathit{r}(Z, Y) &:= P \cup (Z \cap Y){\downarrow}, \\ \mathbf{\mathit{1}} &:= \mathrm{Cm}(A) \end{aligned}$$

for $Z, Y \in \mathcal{H}(\mathbf{A})$.

Proof. First observe that if $a, b \in A$ and $a \neq b$, then from the congruence orderability $M(a) \neq M(b)$. Thus M is injective.

Now we prove that M is surjective. Fix $Z \in \mathcal{H}(\mathbf{A})$. If $Z = \mathrm{Cm}(\mathbf{A})$ then $Z = M(1)$, so assume that $Z \neq \mathrm{Cm}(\mathbf{A})$. Let's take all the equivalence classes on $\underline{L}$, whose intersection with Z is a hyperplane and choose from each of these classes one element which does not belong to Z. More precisely, let $\gamma_0, \ldots, \gamma_n \in \underline{L}$ will be such that for all $i \in \{0, \ldots, n\}$, $\gamma_i^+ \in Z$, but $\gamma_i \notin Z$ and $\gamma_i^+ \neq \gamma_j^+$ for $i, j \in \{0, \ldots, n\}$ such that $i \neq j$. Therefore, $(\overline{Z \cap \gamma_i/_\sim}, \bullet)$ is a hyperplane in $(\overline{\gamma_i/_\sim}, \bullet)$ for all $i \in \{0, \ldots, n\}$. From Lemma 17, for all $i \in \{0, \ldots, n\}$ there exist $a_0, \ldots, a_n \in I(\mathbf{A})$ such that $\overline{M(a_i) \cap \gamma_i/_\sim} = \overline{Z \cap \gamma_i/_\sim}$. Moreover, from Lemma 18, we get that $\psi \in M(a_i)$ for all $\psi \in \mathrm{Cm}(\mathbf{A}) \setminus (\gamma_i/_\sim)$. Thus $Z \subseteq M(a_i)$ and consequently $Z \subseteq \bigcap_{i=0}^n M(a_i)$.

Now, let $\varphi_0, \ldots, \varphi_k \in \widetilde{L} \setminus Z$. Similarly to the above, for all $i \in \{0, \ldots, k\}$, there exist $b_0, \ldots, b_k \in I(\mathbf{A})$ such that $\overline{M(b_i) \cap \varphi_i/_\sim} = \overline{Z \cap \varphi_i/_\sim}$, and $M(b_i) = \mathrm{Cm}(\mathbf{A}) \setminus (\{\varphi_i\}{\downarrow})$. Thus, $\varphi_i \notin Z$ and consequently $Z \subseteq M(b_i)$. Moreover, depending on whether $P \subseteq Z$ or $(\overline{P \cap Z}, \bullet)$ is a hyperplane in $(\overline{P}, \bullet)$, we can choose $c \in I(\mathbf{A}) \cup \{1\}$ such that $M(c) \cap P = Z \cap P$ (if $P \subseteq Z$, we have: $c = 1$) and $L \subseteq M(c)$, so $Z \subseteq M(c)$.

It remains to prove that $Z = M(s)$, where $s := a_0 \cdot \ldots \cdot a_n \cdot b_0 \cdot \ldots \cdot b_k \cdot c$. From the above considerations, it follows that if $u, w \in \mathcal{K} := \{a_0, \ldots, a_n, b_0, \ldots, b_k, c\} \setminus \{1\}$ then $u \nsim w$ (in the sense of [7, p. 613]) and $\mathcal{K} \setminus \{1\}$ is an antichain. Thus, using 5.15 [7] we obtain that

$$\Theta(1, s) = \bigvee_{i=0}^{n} \Theta(1, a_i) \vee \bigvee_{i=0}^{k} \Theta(1, b_i) \vee \Theta(1, c).$$

Hence

$$\bigcap_{i=0}^{n} M(a_i) \cap \bigcap_{i=0}^{k} M(b_i) \cap M(c) = M(s).$$

This yields, $Z \subseteq M(s)$. To see the converse inclusion, fix $\mu \in M(s)$. We need to consider three cases:

1) $\mu \in P$. Then $\mu \in P \cap M(c) = Z \cap P$, and so $\mu \in Z$.

2) $\mu \in \widetilde{L}$. Then $\mu \in \widetilde{L} \cap \bigcap_{i=0}^{k} M(b_i) = \widetilde{L} \setminus (\{\varphi_0\}{\downarrow} \cup \cdots \cup \{\varphi_k\}{\downarrow}) \subseteq Z$. Thus $\mu \in Z$.
3) $\mu \in \underline{L}$. Then $\mu^+ \in \widetilde{L} \cap M(s)$. From 2), we obtain $\mu^+ \in Z$. Thus $\mu/_\sim \subseteq Z$ or $(\overline{\mu/_\sim \cap Z}, \bullet)$ is a hyperplane in $(\overline{\mu/_\sim}, \bullet)$. If $\mu/_\sim \subseteq Z$, we immediately get that $\mu \in Z$. In the second case there exists $i \in \{0, \ldots, n\}$ such that $\mu/_\sim = \gamma_i/_\sim$ and $\overline{M(a_i) \cap \gamma_i/_\sim} = \overline{Z \cap \gamma_i/_\sim}$. Since $\mu \in M(a_i)$, we have $\mu \in Z$.

Finally, we conclude that $Z = M(s)$, thus M is surjective and consequently is bijective. Moreover, M preserve operations $\leftrightarrow$ and r, as standard calculations show. By definition $M(1) = \mathrm{Cm}(\mathbf{A})$. Consequently, M is an isomorphism.

4. Free spectra

Using Theorem 20 we can construct the n-generated free algebra, however, in practice it is still very difficult. In the next sections we show a method of counting the number of elements that belong to the free algebras in $\mathcal{V}(\mathbf{R})$. To make it easier, in this part of the work we show that every $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ is a direct product of two algebras.

4.1. An algebra from $\mathcal{V}(\mathbf{R})$ as a direct product of algebras

Theorem 21
Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ be finite. Then

$$\mathbf{A} \cong \mathbf{A}/_{\bigcap L} \times \mathbf{A}/_{\bigcap P}.$$

Proof. Since $\mathcal{V}(\mathbf{R})$ is a congruence permutable variety, we conclude from Theorem 7.5 [1, p. 92] that it is sufficient to show that

$$\text{(i)} \ \bigcap L \wedge \bigcap P = 0_{\mathbf{A}} \quad \text{and} \quad \text{(ii)} \ \bigcap L \vee \bigcap P = 1_{\mathbf{A}}.$$

For (i) observe that since $\bigcap L \wedge \bigcap P = \bigcap\{\mu : \ \mu \in L \cup P\} = \bigcap\{\mu : \ \mu \in \mathrm{Cm}(\mathbf{A})\}$, thus $\bigcap L \wedge \bigcap P = 0_{\mathbf{A}}$.

To prove (ii) we have to show that $(1, c) \in \bigcap L \vee \bigcap P$ for all $c \in I(\mathbf{A})$. Fix $c \in I(\mathbf{A})$ and suppose that $(1, c) \notin \bigcap P$. Then $(\overline{M(c) \cap P}, \bullet)$ is a hyperplane in $(P, \bullet)$ and from uniqueness from Lemma 18 we get that $\widetilde{L} \subseteq M(c)$. Therefore $\underline{L} \subseteq M(c)$ and consequently $(1, c) \in \bigcap L$.

Note also that if $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ and $\mathbf{A}$ finite, then there exists a natural number n such as $\mathbf{A}/_{\bigcap P} \cong \mathbf{2}^n$. Thus, the following corollary is true.

Corollary 22
Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ be finite. Then there exists $n \in \mathbb{N}$ such that

$$\mathbf{A} \cong \mathbf{A}/_{\bigcap L} \times \mathbf{2}^n.$$

4.2. The construction of the frame

Let $F_{\mathbf{R}}(n)$ be the free n-generated algebra in $\mathcal{V}(\mathbf{R})$ and let $X = \{x_1, x_2, \dots, x_n\}$ be the n-element set of free generators of $F_{\mathbf{R}}(n)$. Let $f_{\mathbf{R}}(n)$ denote the number of elements of $F_{\mathbf{R}}(n)$.

Let us recall that in $\mathrm{Cm}(F_{\mathbf{R}}(n))$ we have two sides: $L = \widetilde{L} \cup \underline{L}$ and P, where $\widetilde{L} = \{\mu \in \mathrm{Cm}(F_{\mathbf{R}}(n)) :\ F_{\mathbf{R}}(n)/\mu \cong \mathbf{2}^{\wedge}\}$, $\underline{L} = \{\mu \in \mathrm{Cm}(F_{\mathbf{R}}(n)) :\ F_{\mathbf{R}}(n)/\mu \cong \mathbf{R}\}$ and $P = \{\mu \in \mathrm{Cm}(F_{\mathbf{R}}(n)) :\ F_{\mathbf{R}}(n)/\mu \cong \mathbf{2}\}$.

The construction of the $\mathrm{Cm}(F_{\mathbf{R}}(n))$ proceeds as follows:

1. Each $\mu \in \mathrm{Cm}(F_{\mathbf{R}}(n))$ is labelled by the set indices $\{i :\ x_i \in X \cap (1/\mu)\} \subseteq \{1, \dots, n\}$.
2. $\widetilde{L}$ (the top layer on the left) has $2^n - 1$ elements labelled by all proper subsets of $\{1, \dots, n\}$, each element forms the one-element equivalence class.
3. P (the top layer on the right) has $2^n - 1$ elements labelled by all proper subsets of $\{1, \dots, n\}$, but in this case all elements form only one equivalence class.
4. If $\mu \in \widetilde{L}$ is labelled by $S \subsetneq \{1, \dots, n\}$, so below μ (i.e. in $\underline{L}$ – the lower layer on the left) there are elements labelled by all proper subsets of S, which form one equivalence class.

This construction is due to the fact that, since $F_{\mathbf{R}}(n)$ is the free algebra, so we can identify every element $\mu \in \mathrm{Cm}(F_{\mathbf{R}}(n))$ with some function f that is a distribution of free generators on $\mathbf{k}$, where $\mathbf{k} \in \{\mathbf{R}, \mathbf{2}, \mathbf{2}^{\wedge}\}$, such that $f^{-1}(\{*\}) \neq \emptyset$. Next, any such function f, can be uniquely extended to a surjective homomorphism $\overline{f}\colon \mathrm{Cm}(F_{\mathbf{R}}(n)) \to \mathbf{k}$. Therefore, $\ker \overline{f} = 1/\mu$ for some $\mu \in \mathrm{Cm}(F_{\mathbf{R}}(n))$ (compare to [10, p. 1347-1350]).

In the figures each dot denotes an element of the frame, while straight lines represent a partial ordering directed upwards.

4.3. $F_{\mathbf{R}}(2)$

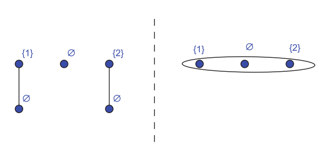

Fig. 1: $\mathbf{Cm}(F_{\mathbf{R}}(2))$.

Observe that $\mathrm{Cm}(F_{\mathbf{R}}(2))$ has 8 elements (Fig. 1): 5 on the left side (each in a separate equivalence class) and 3 on the right side (all in one equivalence class, marked with an ellipse). On the left side we have 18 hereditary sets (all upwards closed sets) and on the right side we have 4 hereditary sets. Summarizing we have $18 \cdot 4$ hereditary sets and so $f_{\mathbf{R}}(2) = 72$.

4.4. $F_{\mathbf{R}}(3)$

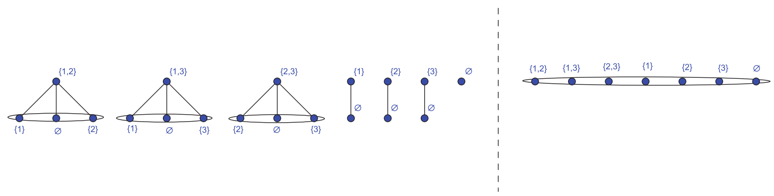

Fig. 2: $\mathbf{Cm}(F_{\mathbf{R}}(3))$.

In this case $\mathrm{Cm}(F_{\mathbf{R}}(3))$ has 26 elements (Fig. 2): 19 on the left side and 7 on the right side. The equivalence classes with more than one element are marked with an ellipse, while each of the other elements form the one-element equivalence classes. On the left side we have 6750 hereditary sets, whereas on the right side we have only 8 hereditary sets. Finally, $f_{\mathbf{R}}(3) = 54000$.

4.5. The formula for the free spectrum

Finally, we give the formula for the free spectrum in $\mathcal{V}(\mathbf{R})$. From Theorem 20 we have $f_{\mathbf{R}}(n) = |\mathcal{H}(F_{\mathbf{R}}(n))|$. Theorem 21 implies

$$f_{\mathbf{R}}(n) = |\mathcal{H}(F_{\mathbf{R}}(n)/_{\bigcap L} \times F_{\mathbf{R}}(n)/_{\bigcap P})|.$$

Then, using the fact that the following function $g\colon \mathcal{H}\Big(F_{\mathbf{R}}(n)/_{\bigcap L} \times F_{\mathbf{R}}(n)/_{\bigcap P}\Big) \to \mathcal{H}\Big(F_{\mathbf{R}}(n)/_{\bigcap L}\Big) \times \mathcal{H}\Big(F_{\mathbf{R}}(n)/_{\bigcap P}\Big)$, given by $g(Z) = (Z \cap L, Z \cap P)$ is a bijection, we get

$$f_{\mathbf{R}}(n) = \Big|\mathcal{H}\Big(F_{\mathbf{R}}(n)/_{\bigcap L}\Big)\Big| \cdot \Big|\mathcal{H}\Big(F_{\mathbf{R}}(n)/_{\bigcap P}\Big)\Big|.$$

Consequently,

$$f_{\mathbf{R}}(n) = |\mathcal{H}(L)| \cdot |\mathcal{H}(P)|.$$

First, we count the number of elements that belong to $\mathcal{H}(P)$. Recall that all elements from P are in one equivalence class. Thus, a subset of $\mathcal{H}(P)$ is a hereditary set iff it is a hyperplane or a whole Boolean group. Since P has $2^n - 1$ hyperplanes, hence

$$|\mathcal{H}(P)| = 2^n.$$

Next, we count the number of elements that belong to $\mathcal{H}(L)$.

Theorem 23
Let $L = \{\mu \in \mathrm{Cm}(F_{\mathbf{R}}(n)) :\ F_{\mathbf{R}}(n)/\mu \cong \mathbf{2}^{\wedge}$ *or* $F_{\mathbf{R}}(n)/\mu \cong \mathbf{R}\}$. *Then*

$$|\mathcal{H}(L)| = \prod_{r=0}^{n-1} (2^r + 1)^{\binom{n}{r}}.$$

Proof. First, we note that if $\mu, \nu \in \widetilde{L}$ and $\mu \neq \nu$ then $\{\mu\}{\downarrow} \cap \{\nu\}{\downarrow} = \emptyset$. Thus, it is enough to count the hereditary sets in $\{\mu\}{\downarrow}$ for $\mu \in \widetilde{L}$, and then multiply results obtained for these sets. Hence, we get $|\mathcal{H}(L)| = \prod_{C \in C(n)}(|T(C)| + 1)$, where $C(n) = P(\{x_1, x_2, \cdots, x_n\}) \setminus \{\{x_1, x_2, \cdots, x_n\}\}$ and $T(C) := \{D :$ D is a proper subset of C$\}$. Similarly to $\mathcal{H}(P)$ we get that $|T(C)| = 2^{|C|}$. Therefore $|\mathcal{H}(L)| = \prod_{C \in C(n)}(2^{|C|} + 1)$.

Note that if $|E| = |F|$ for $E, F \in C(n)$, then $|T(E)| = |T(F)|$. Hence

$$|\mathcal{H}(L)| = \prod_{r=0}^{n-1}(2^r + 1)^{\binom{n}{r}}.$$

Finally, we get the following corollary.

COROLLARY 24

$$f_{\mathbf{R}}(n) = 2^n \prod_{r=0}^{n-1}(2^r + 1)^{\binom{n}{r}}.$$

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