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Metrizable space of multivalued maps

Abstract. In this article we define a metrizable space of multivalued maps. We show that the metric defined in this space is closely related to the homotopy of multivalued maps. Moreover, we study properties of this space and give a few practical applications of the new metric.

1. Introduction

The notion of strongly admissible multivalued maps was introduced by L. Górniewicz (see [1, 2]). Some version of strongly admissible multivalued maps (morphisms) is used to study its properties (see [3, 4, 5]). It is worth mentioning that W. Kryszewski (see [6, 7]) defined morphisms that play an important role in topology. In the paper [8] we applied morphisms to the definition of the homotopy of multivalued mappings. In this article we define morphisms that are applied to the construction of the metric space of multivalued maps.

2. Preliminaries

Let H_* be the *Čech* homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H_*(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q-dimensional *Čech* homology group with compact carriers of X. For a continuous

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map $f\colon X \to Y$, $H_*(f)$ is the induced linear map $f_* = \{f_q\}_{q\geq 0}$, where

$$f_q\colon H_q(X) \to H_q(Y).$$

Throughout this paper all topological spaces are assumed to be metrizable. We say that a compact space X is of finite type if for almost all $q \geq 0$, $H_q(X)$ are trivial and for all $q \geq 0$, $dim\, H_q(X) < \infty$.

Theorem 2.1 (see [1, 2])
Let Y be a compact space of finite type. Then there exists $\varepsilon > 0$ such that for every compact space X and for every two maps $f, g\colon X \to Y$ if $d_Y(f(x), g(x)) < \varepsilon$ for each $x \in X$, then $f_* = g_*$, where d_Y is a metric in Y.

Theorem 2.2 (see [2])
Let X be a metrizable space and let U be an open subset of a normed space $(E, \|\cdot\|)$. In addition, let $i\colon X \to U$ be a compact map. Then for each sufficiently small $\varepsilon > 0$ there exists a finite polyhedron $K_\varepsilon \subset U$ and a map $i_\varepsilon\colon X \to U$ such that

(i) $\|x - i_\varepsilon(x)\| < \varepsilon$ *for all* $x \in X$,

(ii) $i_\varepsilon(X) \subset K_\varepsilon$,

(iii) *the maps* $i, i_\varepsilon\colon X \to U$ *are homotopic.*

A continuous map $f\colon X \to Y$ is called proper if for every compact set $K \subset Y$ the set $f^{-1}(K)$ is nonempty and compact. A proper map $p\colon X \to Y$ is called Vietoris provided for every $y \in Y$ the set $p^{-1}(y)$ is acyclic (in the sense of the Čech homology). The symbol $D(X, Y)$ will denote the set of all diagrams of the form

$$X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y,$$

where $p\colon Z \to X$ denotes a Vietoris map and $q\colon Z \to Y$ denotes a continuous map. Each such diagram will be denoted by (p, q). We recall that the composition of two Vietoris maps is a Vietoris map and if $p\colon X \to Y$ is a Vietoris map then $p_*\colon H_*(X) \to H_*(Y)$ is an isomorphism (see [1]).

Definition 2.3 (see [1, 2])
Let $(p, q) \in D(X, Y)$ and $(r, s) \in D(Y, T)$. The composition of the diagrams

$$X \xleftarrow{\ p\ } Z_1 \xrightarrow{\ q\ } Y \xleftarrow{\ r\ } Z_2 \xrightarrow{\ s\ } T,$$

is called the diagram $(u, v) \in D(X, T)$,

$$X \xleftarrow{\ u\ } Z_1 \triangle_{qr} Z_2 \xrightarrow{\ v\ } T,$$

where

$$Z_1 \triangle_{qr} Z_2 = \{(z_1, z_2) \in Z_1 \times Z_2 :\ q(z_1) = r(z_2)\},$$

$$u = p \circ f_1, \quad v = s \circ f_2,$$

$$Z_1 \xleftarrow{\ f_1\ } Z_1 \triangle_{qr} Z_2 \xrightarrow{\ f_2\ } Z_2,$$

$$f_1(z_1, z_2) = z_1 \text{ (Vietoris map)},\ f_2(z_1, z_2) = z_2 \quad \text{for each } (z_1, z_2) \in Z_1 \triangle_{qr} Z_2.$$

It shall be written

$$(u, v) = (r, s) \circ (p, q).$$

In all other sections it will be assumed that the space X is compact.

3. Pseudometric in the space $D(X,Y)$ and its properties

Let X, Y be metrizable spaces. By the symbol d_X we will denote a metric in the space X. Set

$$\mathbb{C}(T) \equiv \mathbb{C}(T,Y) = \{f\colon T \to Y :\ f \text{ is a continuous map}\},$$

where T is a compact space. In the space $\mathbb{C}(T)$ we have the following metric

$$d_{\mathbb{C}(T)}(f,g) = \sup\{d_Y(f(t),g(t)) :\ t \in T\}.$$

For any proper map $g\colon S \to T$, we observe that

$$d_{\mathbb{C}(S)}(f_1 \circ g, f_2 \circ g) = d_{\mathbb{C}(T)}(f_1, f_2),$$

where $f_1, f_2 \in \mathbb{C}(T)$. Let $(p_1,q_1),(p_2,q_2) \in D(X,Y)$, where

$$X \xleftarrow{\ p_1\ } Z_1 \xrightarrow{\ q_1\ } Y, \quad X \xleftarrow{\ p_2\ } Z_2 \xrightarrow{\ q_2\ } Y.$$

Denote by $\mathfrak{M}$ a set of all metrizable spaces and set

$$\mathbb{V}(X,Y) = \{u\colon X \to Y :\ u \text{ is a Vietoris map}\}.$$

Let $p_i \in \mathbb{V}(Z_i, X)$, $i = 1,2$. We define a set

$$\mathbb{V}(p_1,p_2,X) = \{(\alpha_1,\alpha_2) \in \mathbb{V}(A,Z_1) \times \mathbb{V}(A,Z_2) :\ A \in \mathfrak{M} \text{ and } p_1 \circ \alpha_1 = p_2 \circ \alpha_2\}.$$

Proposition 3.1
For each $p_1 \in \mathbb{V}(Z_1,X)$ and $p_2 \in \mathbb{V}(Z_2,X)$ the set $\mathbb{V}(p_1,p_2,X)$ is nonempty.

Proof. Let $p_1 \in \mathbb{V}(Z_1,X)$, $p_2 \in \mathbb{V}(Z_2,X)$, $A = Z_1 \triangle_{p_1p_2} Z_2$ (see Definition 2.3) and let $\alpha_i \in \mathbb{V}(A,Z_i)$ for $i = 1,2$ be restrictions of projections. Thus, $p_1 \circ \alpha_1 = p_2 \circ \alpha_2$ and the proof is completed.

We define a function $D\colon D(X,Y) \times D(X,Y) \to [0,\infty)$ by the formula

$$D((p_1,q_1),(p_2,q_2)) = \inf\{d_{\mathbb{C}(A)}(q_1 \circ \alpha_1, q_2 \circ \alpha_2) :\ (\alpha_1,\alpha_2) \in \mathbb{V}(p_1,p_2,X)\}.$$

By Proposition 3.1 the above definition is correct.

Proposition 3.2
$D((Id_X,f),(p,q)) = 0$ if and only if $f \circ p = q$.

Proof. Let $D((Id_X,f),(p,q)) = 0$. Then for each $n \in \mathbb{N}$ there exist Vietoris maps $\alpha_n\colon A_n \to X$, $\alpha'_n\colon A_n \to Z$ such that $\alpha_n = p \circ \alpha'_n$ and $d_{\mathbb{C}(A_n)}(f \circ \alpha_n, q \circ \alpha'_n) \le 1/n$. Thus, for each n,

$$d_{\mathbb{C}(Z)}(f \circ p, q) = d_{\mathbb{C}(A_n)}((f \circ p) \circ \alpha'_n, q \circ \alpha'_n) = d_{\mathbb{C}(A_n)}(f \circ \alpha_n, q \circ \alpha'_n) \le 1/n,$$

so $f \circ p = q$. Proof in the opposite direction is obvious.

Proposition 3.3
The function D is a pseudometric in $D(X,Y)$.

Proof. Let $(p_1,q_1)=(p_2,q_2)$. Then it is obvious that $D((p_1,q_1),(p_2,q_2))=0$. It is clear that D is symmetric. Let $(p_1,q_1),(p_2,q_2),(p_3,q_3)\in D(X,Y)$, where

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y,\quad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y,\quad X \xleftarrow{p_3} Z_3 \xrightarrow{q_3} Y.$$

We show that

$$D((p_1,q_1),(p_3,q_3)) \le D((p_1,q_1),(p_2,q_2)) + D((p_2,q_2),(p_3,q_3)).$$

Set $D((p_1,q_1),(p_2,q_2))=\lambda_1$ and $D((p_2,q_2),(p_3,q_3))=\lambda_2$. For each natural number n there exist Vietoris maps $\alpha_n\colon A_n\to Z_1$, $\beta_n\colon A_n\to Z_2$, $\gamma_n\colon B_n\to Z_2$ and $\delta_n\colon B_n\to Z_3$ such that $p_1\circ\alpha_n=p_2\circ\beta_n$, $p_2\circ\gamma_n=p_3\circ\delta_n$ and

$$d_{\mathbb{C}(A_n)}(q_1\circ\alpha_n, q_2\circ\beta_n)\le \lambda_1+1/n,$$
$$d_{\mathbb{C}(B_n)}(q_2\circ\gamma_n, q_3\circ\delta_n)\le \lambda_2+1/n.$$

Fix $n\in\mathbb{N}$ and put

$$C_n=\{(a_n,b_n)\in A_n\times B_n:\ \beta_n(a_n)=\gamma_n(b_n)\}.$$

We denote by $\pi_n\colon C_n\to A_n$ and $\pi'_n\colon C_n\to B_n$ restrictions of projections. Observe that π_n and π'_n are Vietoris maps and $\beta_n\circ\pi_n=\gamma_n\circ\pi'_n$. We have

$$(p_1\circ\alpha_n)\circ\pi_n=(p_2\circ\beta_n)\circ\pi_n=(p_2\circ\gamma_n)\circ\pi'_n=(p_3\circ\delta_n)\circ\pi'_n.$$

Let $\alpha'_n=\alpha_n\circ\pi_n$ and $\beta'_n=\delta_n\circ\pi'_n$ then $p_1\circ\alpha'_n=p_3\circ\beta'_n$ and

$$\begin{aligned}
&d_{\mathbb{C}(C_n)}(q_1\circ\alpha'_n, q_3\circ\beta'_n)\\
&\quad= d_{\mathbb{C}(C_n)}(q_1\circ(\alpha_n\circ\pi_n), q_3\circ(\delta_n\circ\pi'_n))\\
&\quad\le d_{\mathbb{C}(C_n)}(q_1\circ(\alpha_n\circ\pi_n), q_2\circ(\beta_n\circ\pi_n)) + d_{\mathbb{C}(C_n)}(q_2\circ(\beta_n\circ\pi_n), q_3\circ(\delta_n\circ\pi'_n))\\
&\quad= d_{\mathbb{C}(C_n)}((q_1\circ\alpha_n)\circ\pi_n, (q_2\circ\beta_n)\circ\pi_n) + d_{\mathbb{C}(C_n)}((q_2\circ\gamma_n)\circ\pi'_n, (q_3\circ\delta_n)\circ\pi'_n)\\
&\quad= d_{\mathbb{C}(A_n)}(q_1\circ\alpha_n, q_2\circ\beta_n) + d_{\mathbb{C}(B_n)}(q_2\circ\gamma_n, q_3\circ\delta_n)\\
&\quad\le \lambda_1+\lambda_2+2/n.
\end{aligned}$$

Thus

$$D((p_1,q_1),(p_3,q_3))\le\lambda_1+\lambda_2$$

and the proof is completed.

It follows from Proposition 3.2 that D is a not metric in the space $D(X,Y)$.

Let $f,g\colon X\to Y$ be continuous maps.

Proposition 3.4
$D((Id_X,f),(Id_X,g))=d_{\mathbb{C}(X)}(f,g)$.

Proof. Let $r, s\colon Z \to X$ be Vietoris maps such that $r = Id_X \circ r = Id_X \circ s = s$, then

$$d_{\mathbb{C}(Z)}(f \circ r, g \circ s) = d_{\mathbb{C}(Z)}(f \circ r, g \circ r) = d_{\mathbb{C}(X)}(f, g).$$

Let $A \subset X$ be a nonempty set and let

$$O_r(A) = \{x \in X :\ \text{there exists } y \in A \text{ such that } d_X(x, y) < r\},$$

where $r \in \mathbb{R}$, $r > 0$. We denote by d_H the Hausdorff metric, i.e.

$$d_H(A, B) = \inf\{r \geq 0 :\ A \subset O_r(B) \text{ and } B \subset O_r(A)\},$$

where $A, B \subset X$ are nonempty and compact sets.

Proposition 3.5
If $(p_1, q_1), (p_2, q_2) \in D(X, Y)$, then for each $x \in X$

$$d_H(q_1(p_1^{-1}(x)), q_2(p_2^{-1}(x))) \leq D((p_1, q_1), (p_2, q_2)).$$

Proof. Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$, where

$$X \xleftarrow{\ p_1\ } Z_1 \xrightarrow{\ q_1\ } Y, \quad X \xleftarrow{\ p_2\ } Z_2 \xrightarrow{\ q_2\ } Y.$$

and let $D((p_1, q_1), (p_2, q_2)) = \lambda$. For each $n \in \mathbb{N}$ there exist Vietoris maps $\alpha_n\colon A_n \to Z_1$, $\alpha'_n\colon A_n \to Z_2$ such that $p_1 \circ \alpha_n = p_2 \circ \alpha'_n$ and $d_{\mathbb{C}(A_n)}(q_1 \circ \alpha_n, q_2 \circ \alpha'_n) \leq \lambda + 1/n$. Fix $n \in \mathbb{N}$, $x \in X$ and $y_1 \in q_1(p_1^{-1}(x))$. Then $q_1(z_1) = y_1$ and $p_1(z_1) = x$. There exists a point $a_n \in A_n$ such that $\alpha_n(a_n) = z_1$. Let $\alpha'_n(a_n) = z_2$ and $y_2 = q_2(z_2)$, then $z_2 \in p_2^{-1}(x)$ and $y_2 \in q_2(p_2^{-1}(x))$. We have

$$d_Y(y_1, y_2) = d_Y(q_1(\alpha_n(a_n)), q_2(\alpha'_n(a_n))) \leq \lambda + 1/n.$$

Thus, $q_1(p_1^{-1}(x)) \subset O_{\lambda+1/n}(q_2(p_2^{-1}(x)))$. Similarly, we can show, that $q_2(p_2^{-1}(x)) \subset O_{\lambda+1/n}(q_1(p_1^{-1}(x)))$. Therefore, $d_H(q_1(p_1^{-1}(x)), q_2(p_2^{-1}(x))) \leq \lambda$ and the proof is completed.

Let $\varepsilon > 0$ and $(p, q), (r, s) \in D(X, Y)$, where

$$X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y, \quad X \xleftarrow{\ r\ } T \xrightarrow{\ s\ } Y.$$

It is easy to see that

$$(D((p, q), (r, s)) < \varepsilon) \Leftrightarrow ((p \circ \alpha = r \circ \alpha') \text{ and } d_{\mathbb{C}(A)}(q \circ \alpha, s \circ \alpha') < \varepsilon) \tag{1}$$

for some Vietoris maps $\alpha\colon A \to Z$ and $\alpha'\colon A \to T$. We will write

$$(p, q) \sim_{HD} (r, s), \tag{2}$$

if there exist Vietoris maps $\alpha\colon A \to Z$ and $\alpha'\colon A \to T$ such that

$$p \circ \alpha = r \circ \alpha' \text{ and } q \circ \alpha \sim s \circ \alpha',$$

where the symbol $\sim$ denote a homotopy joining the maps $q \circ \alpha$ and $s \circ \alpha'$.

Proposition 3.6
If $(p,q) \sim_{HD} (r,s)$ *then* $q_* \circ p_*^{-1} = s_* \circ r_*^{-1}$.

Proof. Let $(p,q) \sim_{HD} (r,s)$. Then there exist Vietoris maps $\alpha\colon A \to Z$ and $\alpha'\colon A \to T$ such that $p \circ \alpha = r \circ \alpha'$ and $q \circ \alpha \sim s \circ \alpha'$. Thus, $p_* \circ \alpha_* = r_* \circ \alpha'_*$ and $q_* \circ \alpha_* = s_* \circ \alpha'_*$, so $p_* = r_* \circ \beta_*$ and $q_* = s_* \circ \beta_*$, where $\beta_* = \alpha'_* \circ \alpha_*^{-1}$. We have

$$q_* \circ p_*^{-1} = s_* \circ (\beta_* \circ \beta_*^{-1}) \circ r_*^{-1} = s_* \circ r_*^{-1},$$

which completes the proof.

Proposition 3.7
Let $Y \in ANR$ *and let* $(p,q) \in D(X,Y)$. *Then there exists* $\varepsilon > 0$ *such that for every* $(r,s) \in D(X,Y)$ *if* $D((p,q),(r,s)) < \varepsilon$, *then*

$$(p,q) \sim_{HD} (r,s).$$

Proof. There exists an open neighborhood $U \subset E$ of Y and a retraction $f\colon U \to Y$, where E is a normed space. Let $K = q(p^{-1}(X)) \subset U$. The set K is compact, so there exists $\varepsilon > 0$ such that $O_\varepsilon(K) \subset U$. Let $(r,s) \in D(X,Y)$. We assume that $D((p,q),(r,s)) < \varepsilon$. There exist Vietoris maps $\alpha\colon A \to Z$ and $\alpha'\colon A \to T$ such that

$$p \circ \alpha = r \circ \alpha' \text{ and } d_{\mathbb{C}(A)}(q \circ \alpha, s \circ \alpha') < \varepsilon. \tag{3}$$

We have the following diagram

$$\begin{array}{ccccccccc}
X & \xleftarrow{\ p\ } & Z & \xrightarrow{\ q\ } & Y & \xrightarrow{\ i\ } & U & \xrightarrow{\ f\ } & Y \\
\uparrow{\scriptstyle Id_X} & & \uparrow{\scriptstyle \alpha} & & & & & & \\
X & \xleftarrow{p\circ\alpha = r\circ\alpha'} & A & & & & & & \\
\downarrow{\scriptstyle Id_X} & & \downarrow{\scriptstyle \alpha'} & & & & & & \\
X & \xleftarrow{\ r\ } & T & \xrightarrow{\ s\ } & Y & \xrightarrow{\ i\ } & U & \xrightarrow{\ f\ } & Y.
\end{array}$$

We define a map $H\colon A \times [0,1] \to U$ by the formula

$$H(a,t) = (1-t)q(\alpha(a)) + ts(\alpha'(a)) \qquad \text{for each } (a,t) \in A \times [0,1].$$

The map H is well defined (see (3)). Indeed, let $a \in X$ and $t \in [0,1]$. From the Arens-Eels theorem we have

$$\begin{aligned}
\|q(\alpha(a)) - &((1-t)q(\alpha(a)) + ts(\alpha'(a)))\| \\
&= \|(1-t)q(\alpha(a)) + tq(\alpha(a)) - (1-t)q(\alpha(a)) - ts(\alpha'(a))\| \\
&= t\|q(\alpha(a)) - s(\alpha'(a))\| \le \|q(\alpha(a)) - s(\alpha'(a))\| \\
&= d_Y(q(\alpha(a)), s(\alpha'(a))) \le d_{\mathbb{C}(A)}(q \circ \alpha, s \circ \alpha') < \varepsilon,
\end{aligned}$$

where $\|\cdot\|$ is a norm in E. The map $f \circ H$ is a homotopy, joining the maps $q \circ \alpha$ and $s \circ \alpha'$. Thus, $(p,q) \sim_{HD} (r,s)$ and the proof is completed.

Remark 3.8
Let $(p,q),(r,s)\in D(X,\mathbb{S}^n)$. Observe that (see Proposition 3.7) for $\varepsilon=2$ we have

$$(D((p,q),(r,s))<\varepsilon)\Rightarrow((p,q)\sim_{HD}(r,s)).$$

By $\mathbb{S}^n$ denote the sphere with the center of 0 and a radius 1 in the Euclidean space $\mathbb{R}^{n+1}$.

Example 3.9 (see [1])
Let $r\in[0,2)$ and $Z_r=\{(x,y)\in\mathbb{S}^2\times\mathbb{S}^2:\ \|x-y\|\le r\}$. Define Vietoris maps $p_r,p_r^-,q_r,q_r^-\colon Z_r\to\mathbb{S}^2$ by

$$\begin{aligned}p_r(x,y)&=x,\\ p_r^-(x,y)&=-x,\\ q_r(x,y)&=y,\\ q_r^-(x,y)&=-y\end{aligned}$$

for each $(x,y)\in Z_r$. We observe that for any $r\in[0,2)$,

$$p_r\sim q_r\quad\text{and}\quad p_r^-\sim q_r^-.$$

Let $\pi_1,\pi_2\colon(Z_r\triangle Z_r)_{q_rp_r}\to Z_r$ be projection (see Definition 2.3), then

$$p_r\circ\pi_1\sim q_r\circ\pi_1=p_r\circ\pi_2\sim q_r\circ\pi_2$$

and similarly

$$p_r^-\circ\pi_1\sim q_r^-\circ\pi_1=p_r^-\circ\pi_2\sim q_r^-\circ\pi_2.$$

Hence

$$(Id_{\mathbb{S}^2},Id_{\mathbb{S}^2})\sim_{HD}(p_r\circ\pi_1,p_r\circ\pi_1)\sim_{HD}(p_r\circ\pi_1,q_r\circ\pi_2)$$

and

$$(Id_{\mathbb{S}^2},f)\sim_{HD}(p_r\circ\pi_1,p_r^-\circ\pi_1)\sim_{HD}(p_r\circ\pi_1,q_r^-\circ\pi_2),$$

where $f\colon\mathbb{S}^2\to\mathbb{S}^2$ is an antipodal map. From Proposition 3.6 and the fact that $f_*\neq Id_{\mathbb{S}^2*}$ we get

$$(q_r\circ\pi_2)_*\circ(p_r\circ\pi_1)_*^{-1}\neq(q_r^-\circ\pi_2)_*\circ(p_r\circ\pi_1)_*^{-1}.\tag{4}$$

On the other hand, for every $r\in[\sqrt{2},2)$ and for each $x\in\mathbb{S}^2$ we obtain

$$(q_r\circ\pi_2)((p_r\circ\pi_1)^{-1}(x))=(q_r^-\circ\pi_2)((p_r\circ\pi_1)^{-1}(x))=\mathbb{S}^2.\tag{5}$$

We have the diagram

$$\begin{array}{ccccc}
\mathbb{S}^2 & \xleftarrow{p_r\circ\pi_1} & (Z_r\triangle Z_r)_{q_rp_r} & \xrightarrow{q_r\circ\pi_2} & \mathbb{S}^2\\
\uparrow{\scriptstyle Id_{\mathbb{S}^2}} & & \uparrow{\scriptstyle Id_{(Z_r\triangle Z_r)_{q_rp_r}}} & & \\
\mathbb{S}^2 & \xleftarrow{p_r\circ\pi_1} & (Z_r\triangle Z_r)_{q_rp_r} & & \\
\downarrow{\scriptstyle Id_{\mathbb{S}^2}} & & \downarrow{\scriptstyle Id_{(Z_r\triangle Z_r)_{q_rp_r}}} & & \\
\mathbb{S}^2 & \xleftarrow{p_r\circ\pi_1} & (Z_r\triangle Z_r)_{q_rp_r} & \xrightarrow{q_r^-\circ\pi_2} & \mathbb{S}^2.
\end{array}$$

We observe that

$$D((p_r \circ \pi_1, q_r \circ \pi_2), (p_r \circ \pi_1, q_r^- \circ \pi_2)) \leq d_{\mathbb{C}(Z_r \triangle Z_r)_{q_r p_r})}(q_r \circ \pi_2, q_r^- \circ \pi_2) = 2.$$

On the other hand, $D((p_r \circ \pi_1, q_r \circ \pi_2), (p_r \circ \pi_1, q_r^- \circ \pi_2)) \geq 2$ (see (4), Remark 3.8). Thus,

$$D((p_r \circ \pi_1, q_r \circ \pi_2), (p_r \circ \pi_1, q_r^- \circ \pi_2)) = 2, \tag{6}$$

but, for each $x \in \mathbb{S}^2$ (see (5)),

$$d_H((q_r \circ \pi_2)((p_r \circ \pi_1)^{-1}(x)), (q_r^- \circ \pi_2)((p_r \circ \pi_1)^{-1}(x))) = 0.$$

At the end of the section we will formulate and prove a few lemmas which will be used in the following sections. Let $(p_n, q_n), (p, q) \in D(X, Y)$, where

$$X \xleftarrow{p_n} Z_n \xrightarrow{q_n} Y, \quad X \xleftarrow{p} Z \xrightarrow{q} Y,$$

$n = 1, 2, \ldots$. We will write $((p_n, q_n)) \dashrightarrow (p, q)$ if for each $\varepsilon > 0$ there exists n_0 such that for each $n \geq n_0$,

$$D((p_n, q_n), (p, q)) < \varepsilon.$$

Lemma 3.10

$((p_n, q_n)) \dashrightarrow (p, q)$ if and only if there exists a Vietoris map $\beta\colon A \to Z$ such that for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there exists a Vietoris map $\alpha_n\colon A \to Z_n$ for which

$$p_n \circ \alpha_n = p \circ \beta \quad \textit{and} \quad d_{\mathbb{C}(A)}(q_n \circ \alpha_n, q \circ \beta) < \varepsilon.$$

Proof. Fix $k \in \mathbb{N}$. From the assumption it results that for $\varepsilon = 1/k$ there exists $n_k \in \mathbb{N}$ such that for each $n \geq n_k$ there exist Vietoris maps $\alpha_{n,k}\colon A_{n,k} \to Z_n$ and $\beta_{n,k}\colon A_{n,k} \to Z$ such that

$$p_n \circ \alpha_{n,k} = p \circ \beta_{n,k} \quad \text{and} \quad d_{\mathbb{C}(A_{n,k})}(q_n \circ \alpha_{n,k}, q \circ \beta_{n,k}) < \varepsilon. \tag{7}$$

Let

$$A_k = \bigcup_{z \in Z} \prod_{n=n_k}^{\infty} \beta_{n,k}^{-1}(z)$$

and let $\pi_{n,k}\colon A_k \to A_{n,k}$ for $n \geq n_k$ be restrictions of projections. Now set

$$A = \bigcup_{z \in Z} \prod_{k=1}^{\infty} \gamma_k^{-1}(z),$$

where $\gamma_k = \beta_{n_k,k} \circ \pi_{n_k,k} = \beta_{n,k} \circ \pi_{n,k}$ for $k \geq 1$, $n \geq n_k$. Let $\pi_k\colon A \to A_k$ for $k \geq 1$ be restrictions of projections. We define a Vietoris map $\beta\colon A \to Z$ as

$$\beta = \beta_{n_1,1} \circ \pi_{n_1,1} \circ \pi_1 = \beta_{n_k,k} \circ \pi_{n_k,k} \circ \pi_k = \beta_{n,k} \circ \pi_{n,k} \circ \pi_k, \qquad k \geq 1,\ n \geq n_k.$$

To show (7) fix $\varepsilon > 0$ and $k \in \mathbb{N}$ such that $1/k < \varepsilon$. Let $\alpha_n = \alpha_{n,k} \circ \pi_{n,k} \circ \pi_k$ for $n \geq n_k$. We have

$$\begin{aligned} p_n \circ \alpha_n &= p_n \circ \alpha_{n,k} \circ \pi_{n,k} \circ \pi_k = p \circ \beta_{n,k} \circ \pi_{n,k} \circ \pi_k \\ &= p \circ \beta_{n_k,k} \circ \pi_{n_k,k} \circ \pi_k = p \circ \beta_{n_1,1} \circ \pi_{n_1,1} \circ \pi_1 \\ &= p \circ \beta \end{aligned}$$

and

$$\begin{aligned} d_{\mathbb{C}(A)}(q_n \circ \alpha_n, q \circ \beta) &= d_{\mathbb{C}(A)}(q_n \circ \alpha_{n,k} \circ \pi_{n,k} \circ \pi_k, q \circ \beta_{n_1,1} \circ \pi_{n_1,1} \circ \pi_1) \\ &= d_{\mathbb{C}(A)}(q_n \circ \alpha_{n,k} \circ \pi_{n,k} \circ \pi_k, q \circ \beta_{n_k,k} \circ \pi_{n_k,k} \circ \pi_k) \\ &= d_{\mathbb{C}(A)}(q_n \circ \alpha_{n,k} \circ \pi_{n,k} \circ \pi_k, q \circ \beta_{n,k} \circ \pi_{n,k} \circ \pi_k) \\ &= d_{\mathbb{C}(A_{n,k})}(q_n \circ \alpha_{n,k}, q \circ \beta_{n,k}) \\ &< 1/k < \varepsilon. \end{aligned}$$

Proof in the opposite direction is obvious.

Lemma 3.11
$((p_n, q_n)) \dashrightarrow (p, q)$ if and only if there exist Vietoris maps $\beta\colon A \to Z$, $\alpha_n\colon A \to Z_n$, $n = 1, 2, \ldots$ such that

$$p_n \circ \alpha_n = p \circ \beta \quad \textit{and} \quad \lim_{n\to\infty} q_n \circ \alpha_n = q \circ \beta. \tag{8}$$

Proof. From Lemma 3.10 we get a Vietoris map $\beta\colon A \to Z$ such that for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there exist Vietoris maps $\alpha_n\colon A \to Z_n$ for which

$$p_n \circ \alpha_n = p \circ \beta \quad \text{and} \quad d_{\mathbb{C}(A)}(q_n \circ \alpha_n, q \circ \beta) < \varepsilon.$$

Let $\varepsilon = 1/k$, $k \in \mathbb{N}$. There exist Vietoris maps $\alpha_{n,k}\colon A \to Z_n$ such that

$$p_n \circ \alpha_{n,k} = p \circ \beta \quad \text{and} \quad d_{\mathbb{C}(A)}(q_n \circ \alpha_{n,k}, q \circ \beta) < 1/k$$

for each $n \geq n_k$. We can assume that $1 = n_1 < n_2 < \ldots < n_k < \ldots$. We define

$$\alpha_n = \begin{cases} \alpha_{n,1} & \text{for } n_1 \leq n < n_2, \\ \vdots & \\ \alpha_{n,k} & \text{for } n_k \leq n < n_{k+1}, \\ \vdots & \end{cases}$$

$k = 2, 3, \ldots$. We observe that the sequence $\alpha_n\colon A \to Z_n$, $n = 1, 2, \ldots$ satisfies condition (8). Proof in the opposite direction is obvious.

Lemma 3.12
Let $\lambda_n > 0$, $n = 1, 2, \ldots$. For each $n \in \mathbb{N}$,

$$D((p_n, q_n), (p, q)) < \lambda_n$$

if and only if there exist a space T, a Vietoris map $u\colon T \to Z$ and for each n, a Vietoris map $v_n\colon T \to Z_n$ such that

$$p_n \circ v_n = p \circ u \quad \textit{and} \quad d_{\mathbb{C}(T)}(q_n \circ v_n, q \circ u) < \lambda_n.$$

Proof. From the assumption there exist Vietoris maps $r_n\colon T_n \to Z$ and $s_n\colon T_n \to Z_n$ such that $p_n \circ s_n = p \circ r_n$ and $d_{\mathbb{C}(T_n)}(q_n \circ s_n, q \circ r_n) < \lambda_n$, $n = 1, 2, \ldots$. Let

$$T = \bigcup_{z\in Z} \prod_{n=1}^{\infty} r_n^{-1}(z) \subset \prod_{n=1}^{\infty} T_n$$

and let, for each n, $\pi_n\colon T \to T_n$ be a restriction of a projection map. Notice that, for each n, π_n is a Vietoris map and

$$r_n \circ \pi_n = r_{n+1} \circ \pi_{n+1}.$$

Setting $u = r_1 \circ \pi_1$ and $v_n = s_n \circ \pi_n$ we have for each n,

$$p_n \circ v_n = (p_n \circ s_n) \circ \pi_n = (p \circ r_n) \circ \pi_n = p \circ (r_1 \circ \pi_1) = p \circ u$$

and

$$\begin{aligned} d_{\mathbb{C}(T)}(q_n \circ v_n, q \circ u) &= d_{\mathbb{C}(T)}(q_n \circ (s_n \circ \pi_n), q \circ (r_1 \circ \pi_1)) \\ &= d_{\mathbb{C}(T)}(q_n \circ (s_n \circ \pi_n), q \circ (r_n \circ \pi_n)) \\ &= d_{\mathbb{C}(T)}((q_n \circ s_n) \circ \pi_n, (q \circ r_n) \circ \pi_n) \\ &= d_{\mathbb{C}(T_n)}(q_n \circ s_n, q \circ r_n) < \lambda_n. \end{aligned}$$

Proof in the opposite direction is obvious.

4. *Dist*-morphisms

Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$. We define a relation in $D(X, Y)$ by the formula:

$$(p_1, q_1) \sim_{dist} (p_2, q_2) \Leftrightarrow D((p_1, q_1), (p_2, q_2)) = 0. \tag{9}$$

From Proposition 3.3 we obtain.

Proposition 4.1
The relation (9) *is an equivalence relation in* $D(X, Y)$.

The set of all equivalence classes of the above relation will be denoted by

$$M_{dist}(X, Y) = D(X, Y)_{/\sim_{dist}}. \tag{10}$$

The elements of the space $M_{dist}(X, Y)$ will be called *dist*-morphisms.

Definition 4.2
For any $\varphi_{dist} \in M_{dist}(X, Y)$, the set $\varphi(x) = q(p^{-1}(x))$, where $\varphi_{dist} = [(p, q)]_{dist}$ is called the image of the point x in the *dist*-morphism φ_{dist}.

We observe that from Proposition 3.5

$$((p, q) \sim_{dist} (r, s)) \Rightarrow (q(p^{-1}(x)) = s(r^{-1}(x)) \text{ for each } x \in X).$$

Thus, it follows that Definition 4.2 is correct. We will write

$$\varphi\colon X \to_{\mathbb{D}} Y$$

if it is a multivalued map determined by the *dist*-morphism $\varphi_{dist} = [(p,q)]_{dist}$, where $(p,q) \in D(X,Y)$. We recall that a multivalued map $\varphi\colon X \multimap Y$ is called strongly admissible (see [1]) provided there exists a diagram $(p,q) \in D(X,Y)$ such that $q(p^{-1}(x)) = \varphi(x)$ for each $x \in X$. From Example 3.9 we get

Example 4.3
Let $\varphi\colon \mathbb{S}^2 \multimap \mathbb{S}^2$ be a multivalued map given by the formula $\varphi(x) = \mathbb{S}^2$ for each $x \in \mathbb{S}^2$. Observe that φ is strongly admissible (see (5)). Let $\varphi_1, \varphi_2\colon \mathbb{S}^2 \to_{\mathbb{D}} \mathbb{S}^2$ be multivalued maps determined by

$$(\varphi_1)_{dist} = [(p_r \circ \pi_1, q_r \circ \pi_2)]_{dist} \quad \text{and} \quad (\varphi_2)_{dist} = [(p_r \circ \pi_1, q_r^- \circ \pi_2)]_{dist}$$

respectively, where $r \in [\sqrt{2}, 2)$. From (6) it results that $\varphi_1 \neq \varphi_2$, but for each $x \in \mathbb{S}^2$ we have $\varphi_1(x) = \varphi_2(x) = \varphi(x) = \mathbb{S}^2$ (see (5)).

Proposition 3.7 implies the following result.

Proposition 4.4
Let $Y \in ANR$ and $(p,q),(r,s) \in D(X,Y)$. If $D((p,q),(r,s)) = 0$ then $(p,q) \sim_{HD} (r,s)$.

Definition 4.5
Let $Y \in ANR$ and $\varphi, \psi\colon X \to_{\mathbb{D}} Y$ be maps determined by $\varphi_{dist} = [(p,q)]_{dist}$ and $\psi_{dist} = [(r,s)]_{dist}$ respectively, where $(p,q),(r,s) \in D(X,Y)$. We will say that φ and ψ are homotopic (we will write $\varphi \sim_{\mathbb{D}} \psi$) if $(p,q) \sim_{HD} (r,s)$.

We denote by $M_{\mathbb{D}}(X,Y) \approx M_{dist}(X,Y)$ (see (10)) a set of all multivalued maps $\varphi\colon X \to_{\mathbb{D}} Y$.

Lemma 4.6
The relation $\sim_{HD}$ (see (2)) is transitive in the set $D(X,Y)$.

Proof. Let $(p,q),(r,s),(u,v) \in D(X,Y)$, where

$$X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y, \quad X \xleftarrow{\ r\ } T \xrightarrow{\ s\ } Y, \quad X \xleftarrow{\ u\ } P \xrightarrow{\ v\ } Y.$$

There exist Vietoris maps $\alpha\colon A \to Z$, $\beta\colon A \to T$, $\gamma\colon B \to T$, $\delta\colon B \to P$ such that

$$p \circ \alpha = r \circ \beta, \quad q \circ \alpha \sim s \circ \beta \quad \text{and} \quad r \circ \gamma = u \circ \delta, \quad s \circ \gamma \sim v \circ \delta.$$

Let (see Definition 2.3)

$$A \xleftarrow{\ \pi_A\ } A \triangle_{\beta\gamma} B \xrightarrow{\ \pi_B\ } B,$$

where π_A, π_B are restrictions of projections and

$$\beta \circ \pi_A = \gamma \circ \pi_B.$$

It is clear that $\alpha \circ \pi_A$ and $\delta \circ \pi_B$ are Vietoris maps. We have

$$p \circ (\alpha \circ \pi_A) = r \circ (\beta \circ \pi_A) = r \circ (\gamma \circ \pi_B) = u \circ (\delta \circ \pi_B)$$

and

$$q \circ (\alpha \circ \pi_A) \sim s \circ (\beta \circ \pi_A) = s \circ (\gamma \circ \pi_B) \sim v \circ (\delta \circ \pi_B).$$

This completes the proof.

PROPOSITION 4.7
If $Y \in ANR$ then the homotopy $\sim_{\mathbb{D}}$ is an equivalence relation in $M_{\mathbb{D}}(X, Y)$.

Proof. From Proposition 4.4 the relation $\sim_{\mathbb{D}}$ is reflexive. It is obvious that it is symmetrical. We show that $\sim_{\mathbb{D}}$ is transitive. Let $\varphi, \psi, \eta \in M_{\mathbb{D}}(X, Y)$ be such that

$$\varphi \sim_{\mathbb{D}} \psi \quad \text{and} \quad \psi \sim_{\mathbb{D}} \eta.$$

From Definition 4.5 there exist $(p, q) \in \varphi_{dist}$, $(r, s) \in \psi_{dist}$, $(r', s') \in \psi_{dist}$ and $(u, v) \in \eta_{dist}$ such that

$$(p, q) \sim_{HD} (r, s) \quad \text{and} \quad (r', s') \sim_{HD} (u, v).$$

We must show that $(p, q) \sim_{HD} (u, v)$. From Proposition 4.4 we obtain

$$(r, s) \sim_{HD} (r', s').$$

From Lemma 4.6 $(p, q) \sim_{HD} (u, v)$ and the proof is completed.

Let $\varphi\colon X \to_{\mathbb{D}} Y$ be a map determined by $\varphi_{dist} = [(p, q)]_{dist}$, $Y \in ANR$. We define

$$\varphi_* = q_* \circ p_*^{-1}. \tag{11}$$

From Proposition 4.4 and Proposition 3.6 it results that definition (11) is correct.

THEOREM 4.8
The space $M_{\mathbb{D}}(X, Y)$ is metrizable.

Proof. Let $\mathbb{D}\colon M_{\mathbb{D}}(X, Y) \times M_{\mathbb{D}}(X, Y) \to [0, \infty)$ be a map given by the formula

$$\mathbb{D}(\varphi, \psi) \equiv \mathbb{D}(\varphi_{dist}, \psi_{dist}) = D((p, q), (r, s)),$$

where $\varphi, \psi \in M_{\mathbb{D}}(X, Y)$ are determined by $\varphi_{dist} = [(p, q)]_{dist}$ and $\psi_{dist} = [(r, s)]_{dist}$, respectively and let $(p, q), (r, s) \in D(X, Y)$. The map $\mathbb{D}$ is well defined. Indeed, let $(p_1, q_1) \sim_{dist} (p, q)$ and $(r_1, s_1) \sim_{dist} (r, s)$. We show that

$$D((p_1, q_1), (r_1, s_1)) = D((p, q), (r, s)).$$

We have

$$\begin{aligned} D((p_1, q_1), (r_1, s_1)) &\le D((p_1, q_1), (p, q)) + D((p, q), (r, s)) + D((r, s), (r_1, s_1)) \\ &= D((p, q), (r, s)) \end{aligned}$$

and similarly

$$D((p, q), (r, s)) \le D((p_1, q_1), (r_1, s_1)).$$

It is easy to show that $\mathbb{D}$ satisfies the metric conditions (see Proposition 3.3) and the proof is completed.

5. Properties of the space $M_{\mathbb{D}}(X, Y)$

We denote by

$$B(\varphi, r) = \{\psi \in M_{\mathbb{D}}(X, Y) :\ \mathbb{D}(\varphi, \psi) < r\}$$

an open ball in $M_{\mathbb{D}}(X, Y)$ with the center of φ and a radius $r > 0$.

Proposition 5.1
The set $\mathbb{C}(X, Y)$ is closed in $M_{\mathbb{D}}(X, Y)$.

Proof. We show that for each $\varphi \in (M_{\mathbb{D}}(X, Y) \backslash \mathbb{C}(X, Y))$ there exists $r > 0$ such that

$$B(\varphi, r) \cap \mathbb{C}(X, Y) = \emptyset.$$

Assume the contrary, that there exists $\varphi \in (M_{\mathbb{D}}(X, Y) \backslash \mathbb{C}(X, Y))$ such that for each $\varepsilon > 0$,

$$B(\varphi, \varepsilon) \cap \mathbb{C}(X, Y) \neq \emptyset.$$

There exists a point $x_0 \in X$ such that $diam(\varphi(x_0)) = r > 0$. Let $\varepsilon = r/3$. From the assumption there exists $f \in \mathbb{C}(X, Y)$ such that $f \in B(\varphi, \varepsilon)$. By Proposition 3.5 we get that for each $x \in X$,

$$d_H(\varphi(x), f(x)) < \varepsilon.$$

Thus, $\varphi(x_0) \subset O_\varepsilon(f(x_0))$, but it is a contradiction and the proof is completed.

Proposition 5.2
Let $Y \in ANR$ be a compact space. Then $\mathbb{C}(X, Y) \subset M_{\mathbb{D}}(X, Y)$ is a boundary set.

Proof. Let $f \in \mathbb{C}(X, Y)$. We show that for each $\varepsilon > 0$,

$$B(f, \varepsilon) \cap (M_{\mathbb{D}}(X, Y) \setminus \mathbb{C}(X, Y)) \neq \emptyset.$$

There exists a sequence $(f_n) \subset \mathbb{C}(X, Y)$ such that

$$\lim_{n\to\infty} f_n = f \quad \text{and} \quad f_n \neq f \text{ for each } n \in \mathbb{N}. \tag{12}$$

The space $Y \in ANR$, so there exist an open neighborhood $V \subset Q$ of Y and a retraction $r\colon \overline{V} \to Y$. Let $\varepsilon = dist(Y, Q \setminus V)$. The map r is uniformly continuous, so there exists $0 < \delta < \varepsilon$ such that for each $x, y \in \overline{V}$ we have

$$(d_Q(x, y) < \delta) \Rightarrow (d_Y(r(x), r(y)) < \varepsilon). \tag{13}$$

There exists n_0 such that for each $n \geq n_0$, $d_{\mathbb{C}(X)}(f_n, f) < \delta$. Let $g = f_{n_0}$. We define a map $s\colon X \times [0, 1] \to \overline{V}$ by the formula

$$s(x, t) = (1 - t)f(x) + tg(x) \qquad \text{for each } (x, t) \in X \times [0, 1].$$

Let $h\colon X \times [0, 1] \to Y$ be a map given by

$$h = r \circ s.$$

We have the following diagram

$$\begin{array}{ccccc} X & \xleftarrow{p} & X \times [0,1] & \xrightarrow{h} & Y \\ \Big\uparrow{\scriptstyle Id_X} & & \Big\uparrow{\scriptstyle Id_{X\times[0,1]}} & & \\ X & \xleftarrow{p} & X \times [0,1] & & \\ \Big\downarrow{\scriptstyle Id_X} & & \Big\downarrow{\scriptstyle p} & & \\ X & \xleftarrow{Id_X} & X & \xrightarrow{f} & Y, \end{array}$$

where $p\colon X \times [0,1] \to X$ is a projection. It is obvious that p is a Vietoris map. We observe that $d_{\mathbb{C}(X\times[0,1],\overline{V})}(s, f \circ p) < \delta$. From (13) we get $d_{\mathbb{C}(X\times[0,1])}(h, f \circ p) < \varepsilon$. Thus,

$$\mathbb{D}((Id_X, f), (p, h)) < \varepsilon.$$

Let $\varphi\colon X \to_{\mathbb{D}} Y$ be a multivalued map determined by $\varphi_{dist} = [(p,h)]_{dist}$. There exists a point $x_0 \in X$ such that $g(x_0) \neq f(x_0)$ (see (12)). It is obvious that $f(x_0) = r(f(x_0)) \in h(p^{-1}(x_0))$ and $g(x_0) = r(g(x_0)) \in h(p^{-1}(x_0))$, so $\varphi \notin \mathbb{C}(X,Y)$ and $\varphi \in B(f,\varepsilon)$ and the proof is completed.

Proposition 5.3

Let Y be a locally compact space. If Y is contractible then $M_{\mathbb{D}}(X,Y)$ is a contractible space.

Proof. Let Y be a contractible space and let $y_0 \in Y$. There exists a map $h\colon Y \times [0,1] \to Y$ such that

$$h(y,0) = y \quad \text{and} \quad h(y,1) = y_0 \qquad \text{for each } y \in Y.$$

Let

$$X \xleftarrow{p} Z \xrightarrow{q_0} Y,$$

where $p\colon Z \to X$ is a Vietoris map and $q_0\colon Z \to Y$ is a constant map, i.e. $q_0(z) = y_0$ for each $z \in Z$. We observe that $\varphi_0\colon X \to_{\mathbb{D}} Y$, determined by $(\varphi_0)_{dist} = [(p,q_0)]_{dist}$, is constant, i.e. $\varphi_0(x) = y_0$ for each $x \in X$. Define a homotopy $H\colon M_{\mathbb{D}}(X,Y) \times [0,1] \to M_{\mathbb{D}}(X,Y)$ by the formula

$$H([(p,q)]_{dist}, t) = [(p, h(q,t))]_{dist} \qquad \text{for each } [(p,q)]_{dist} \in M_{\mathbb{D}}(X,Y) \text{ and } t \in [0,1],$$

where for every $t \in [0,1]$ and $q\colon Z \to Y$ the map $h(q,t)\colon Z \to Y$ is given by

$$h(q,t)(z) = h(q(z),t) \qquad \text{for each } z \in Z.$$

The map H is well defined. Indeed, let $t \in [0,1]$ and let $(r,s) \sim_{dist} (p,q)$, where

$$X \xleftarrow{p} Z \xrightarrow{q} Y, \quad X \xleftarrow{r} T \xrightarrow{s} Y.$$

From Lemma 3.12 there exist Vietoris maps $\alpha_n\colon A \to Z$, $\beta\colon A \to T$ such that

$$p \circ \alpha_n = r \circ \beta \quad \text{and} \quad d_{\mathbb{C}(A)}(q \circ \alpha_n, s \circ \beta) < 1/n, \qquad n = 1, 2, \ldots.$$

Let $K = s(\beta(A))$. It is clear that $K \subset Y$ is compact. The space Y is locally compact, so there exists $\lambda > 0$ such that $\overline{O_\lambda(K)}$ is compact. There exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ and for each $a \in A$, $q(\alpha_n(a)) \in \overline{O_\lambda(K)}$. The map $h(\cdot, t)\colon \overline{O_\lambda(K)} \to Y$ is uniformly continuous. Thus,

$$(r, h(s,t)) \sim_{dist} (p, h(q,t)).$$

Similarly, we show that H is continuous (see Lemma 3.11). It is clear that for each $[(p,q)]_{dist} \in M_{\mathbb{D}}(X,Y)$,

$$H([(p,q)]_{dist}, 0) = [(p,q)]_{dist} \quad \text{and} \quad H([(p,q)]_{dist}, 1) = [(p,q_0)]_{dist}$$

and the proof is completed.

6. Applications of $\mathbb{D}$-metric

Let $\varphi_n, \varphi\colon X \to_{\mathbb{D}} Y$ be multivalued maps determined by $(\varphi_n)_{dist} = [(p_n, q_n)]_{dist}$ and $\varphi_{dist} = [(p,q)]_{dist}$, respectively, where

$$X \xleftarrow{\;p\;} Z \xrightarrow{\;q\;} Y, \quad X \xleftarrow{\;p_n\;} Z_n \xrightarrow{\;q_n\;} Y$$

and $n = 1, 2, \ldots$. Let $\varphi\colon X \to_{\mathbb{D}} X$ be a multivalued map determined by $\varphi_{dist} = [(p,q)]_{dist}$. We recall that φ has a fixed point (we write $Fix(\varphi) \neq \emptyset$) if there exists $x_0 \in X$ such that $x_0 \in \varphi(x_0)$. We observe that x_0 is a fixed point of φ if and only if there exists $z_0 \in Z$ such that $p(z_0) = q(z_0) = x_0$.

Proposition 6.1

Let X be a compact space and let $\varphi\colon X \to_{\mathbb{D}} X$ be a multivalued map determined by $\varphi_{dist} = [(p,q)]_{dist}$. If for each $\varepsilon > 0$ there exists a multivalued map $\varphi_\varepsilon\colon X \to_{\mathbb{D}} X$ such that $Fix(\varphi_\varepsilon) \neq \emptyset$ and $\mathbb{D}(\varphi_\varepsilon, \varphi) < \varepsilon$, then $Fix(\varphi) \neq \emptyset$.

Proof. From the assumption, for each n there exists $\varphi_n\colon X \to_{\mathbb{D}} X$ determined by $(\varphi_n)_{dist} = [(p_n, q_n)]_{dist}$ such that $Fix(\varphi_n) \neq \emptyset$ and $\mathbb{D}(\varphi_n, \varphi) < 1/n$. From Lemma 3.12 there exist a space T, a Vietoris map $u\colon T \to Z$ and for each n a Vietoris map $v_n\colon T \to Z_n$ such that

$$p_n \circ v_n = p \circ u \quad \text{and} \quad d_{\mathbb{C}(T)}(q_n \circ v_n, q \circ u) < 1/n.$$

It is clear that T is a compact space. Let $(x_n) \subset X$ be a sequence of fixed points such that $x_n \in \varphi_n(x_n)$ for each n. Thus, for each n there exists $t_n \in T$ such that

$$p(u(t_n)) = p_n(v_n(t_n)) = q_n(v_n(t_n)) = x_n$$

and

$$d_X(p(u(t_n)), q(u(t_n))) = d_X(q_n(v_n(t_n)), q(u(t_n))) < 1/n.$$

We can assume that t_n is convergent to $t_0 \in T$. Thus, $p(u(t_0)) = q(u(t_0))$ and φ has a fixed point, this ends the proof.

From Proposition 3.7 (see Definition 4.5) we get.

Proposition 6.2
Let $Y \in ANR$ and let $\varphi \in M_{\mathbb{D}}(X,Y)$. Then there exists $\varepsilon > 0$ such that for every $\psi \in M_{\mathbb{D}}(X,Y)$ if $\mathbb{D}(\varphi,\psi) < \varepsilon$, then $\varphi \sim_{\mathbb{D}} \psi$.

Proposition 6.3
Let E be a normed space and let $\varphi\colon T \to_{\mathbb{D}} U$ be a compact multivalued map ($\overline{\varphi(T)} \subset U$ is compact) determined by $\varphi_{dist} = [(p,q)]_{dist}$, where $U \subset E$ is an open set. Then for each sufficiently small $\varepsilon > 0$ there exist a finite polyhedron $K_\varepsilon \subset U$ and a multivalued map $\varphi_\varepsilon\colon T \to_{\mathbb{D}} U$ such that the following conditions are satisfied

$$\mathbb{D}(\varphi_\varepsilon,\varphi) < \varepsilon, \tag{14}$$
$$\varphi_\varepsilon(T) \subset K_\varepsilon, \tag{15}$$
$$\varphi_\varepsilon \sim_{\mathbb{D}} \varphi. \tag{16}$$

Proof. Let $K = \overline{\varphi(T)} = \overline{q(p^{-1}(T))} \subset U$ and let $i\colon K \hookrightarrow U$ be an inclusion. We have

$$T \xleftarrow{\ p\ } Z \xrightarrow{\ \widetilde{q}\ } K \xrightarrow{\ i\ } U,$$

where $\widetilde{q}(z) = q(z)$ for each $z \in Z$. It follows from Theorem 2.2 that for sufficiently small $\varepsilon > 0$ there exists a finite polyhedron $K_\varepsilon \subset U$ and a continuous function $i_\varepsilon\colon K \to U$ such that the conditions of Theorem 2.2 are satisfied. Let $\varphi_\varepsilon\colon T \to_{\mathbb{D}} U$ be a multivalued map determined by $(\varphi_\varepsilon)_{dist} = [(p, i_\varepsilon \circ \widetilde{q})]_{dist}$. Then the conditions (14)–(16) are satisfied and the proof is completed.

Proposition 6.4
Let Y be a compact space of finite type. Then there exists $\varepsilon > 0$ such that for every compact space X and for every two maps $\varphi,\psi \in M_{\mathbb{D}}(X,Y)$, if $\mathbb{D}(\varphi,\psi) < \varepsilon$, then $\varphi_* = \psi_*$.

Proof. We take $\varepsilon_1 > 0$ from Theorem 2.1 and fix arbitrary $\varepsilon < \varepsilon_1$. Let φ and ψ be determined by $\varphi_{dist} = [(p,q)]_{dist}$ and $\psi_{dist} = [(r,s)]_{dist}$, respectively, where

$$X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y, \quad X \xleftarrow{\ r\ } T \xrightarrow{\ s\ } Y.$$

Assume that

$$\mathbb{D}(\varphi,\psi) = D((p,q),(r,s)) < \varepsilon.$$

There exist Vietoris maps (see (1)) $\alpha\colon A \to Z$, $\alpha'\colon A \to T$ such that

$$p \circ \alpha = r \circ \alpha' \quad \text{and} \quad d_{\mathbb{C}(A)}(q \circ \alpha, s \circ \alpha') < \varepsilon.$$

Hence and from Theorem 2.1,

$$p_* \circ \alpha_* = r_* \circ \alpha'_* \quad \text{and} \quad q_* \circ \alpha_* = s_* \circ \alpha'_*.$$

Let $u_* = p_* \circ \alpha_* = r_* \circ \alpha'_*$ and $v_* = q_* \circ \alpha_* = s_* \circ \alpha'_*$. We have

$$v_* \circ u_*^{-1} = (q_* \circ \alpha_*) \circ (p_* \circ \alpha_*)^{-1} = (q_* \circ \alpha_*) \circ (\alpha_*^{-1} \circ p_*^{-1}) = q_* \circ p_*^{-1}$$

and similarly

$$v_* \circ u_*^{-1} = (s_* \circ \alpha'_*) \circ (r_* \circ \alpha'_*)^{-1} = (s_* \circ \alpha'_*) \circ (\alpha'^{-1}_* \circ r_*^{-1}) = s_* \circ r_*^{-1}.$$

Thus,

$$\varphi_* = q_* \circ p_*^{-1} = s_* \circ r_*^{-1} = \psi_*$$

and the proof is completed.

7. Conclusion

Dist-morphisms have many interesting properties and applications. They constitute a very good tool for studying the properties of multivalued maps. They have been used for the construction of the metric space of multivalued maps (see Theorem 4.8). In section 5 a few properties of such a space were given. In section 6 on the other hand, a few practical applications of $\mathbb{D}$-metrics in topology were given.

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