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A determinantal formula for circuits of integer lattices

Abstract. Let L be a not necessarily saturated lattice in $\mathbb{Z}^n$ with a defining matrix B. We explicitly compute the set of circuits of L in terms of maximal minors of B. This has a variety of applications from toric to tropical geometry, from Gröbner to Graver bases, and from linear to binomial ideals.

1. Introduction

The notion of circuit originates from matroid theory (see [4]) and it appears in diverse areas of mathematics including linear spaces, linear ideals [9, p. 3], toric ideals ([9, p. 33], and [3]), toric varieties [8], tropical geometry [2], convex analysis [5], integer lattices, lattice ideals [6], Graver bases and Gröbner bases [6, 9]. The definition often varies partially from each context to the other. A general form of a circuit is when it lives in a general integer lattice as in [6, Definition 2.2]. In this paper we want to compute such circuits explicitly.

Let L be a lattice in $\mathbb{Z}^n$ of dimension m, and let B be a defining matrix of L. By definition, B is an integer $n \times m$ matrix of rank m whose columns generate L as a lattice. The defining matrix B is not unique, but it is unique up to action of the group of unimodular matrices, $\mathrm{GL}_m(\mathbb{Z})$, consisting of all $m \times m$ integer matrices whose determinants are ± 1. This uniqueness means that if B' is another defining matrix of L, then there exists a matrix $U \in \mathrm{GL}_m(\mathbb{Z})$ such that $B' = BU$ [7, Corollary 4.3a].

According to [6, Definition 2.2], a non-zero element $\mathbf{u} := (u_1, \dots, u_n) \in L$ is said to be a circuit if the support of $\mathbf{u}$, i.e. $\mathrm{supp}(\mathbf{u}) := \{i : \ u_i \neq 0\}$, is minimal

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with respect to inclusion and $\frac{1}{d}\mathbf{u} \notin L$ for any positive integer $d \neq 1$. By [6, Remark 2.5, and Proposition 2.6], the set of circuits of L is finite and is a special generating set of L. We present an explicit formula for a circuit of L in terms of maximal minors of the matrix B.

We recall some useful definitions and notation. For arbitrary integer sequences $1 \leq i_1, \dots, i_s \leq n$ and $1 \leq j_1, \dots, j_t \leq m$, we denote by $B[i_1, \dots, i_s | j_1, \dots, j_t]$, the matrix obtained from B by choosing those rows and columns of B which correspond to $i_1, \dots, i_s$ and $j_1, \dots, j_t$, respectively. We define

$$D_B[i_1, \dots, i_m] := \det B[i_1, \dots, i_m | 1, \dots, m].$$

For the integers k_i with $1 \leq k_1 < \dots < k_{m-1} \leq n$, we consider the vector

$$\mathbf{C}^B_{k_1,\dots,k_{m-1}} := D_B[k_1, \dots, k_{m-1}, 1]\mathbf{e}_1 + \dots + D_B[k_1, \dots, k_{m-1}, n]\mathbf{e}_n,$$

where $\mathbf{e}_i$ are standard vectors of $\mathbb{Z}^n$. The vector $\mathbf{C}^B_{k_1,\dots,k_{m-1}}$ is the main ingredient of the formula. We denote by $\beta^B_{k_1,\dots,k_{m-1}}$ the greatest common divisor of the values $D_B[k_1, \dots, k_{m-1}, i]$ for $i = 1, \dots, n$, and we assume it to be positive. In the special case, when $m = 1$, as the multi-index is empty, we set

$$\mathbf{C}^B_{k_1,\dots,k_{m-1}} := D_B[1]\mathbf{e}_1 + \dots + D_B[n]\mathbf{e}_n.$$

For the lattice L, the lattice $\widetilde{L}$ consisting of all $\mathbf{u} \in \mathbb{Z}^n$ for which $r\mathbf{u} \in L$ for some positive integer r is called the saturation of L. Clearly, we have $\widetilde{L} \supseteq L$, and if the equality occurs, we say that L is saturated. Since $\widetilde{L}$ is saturated, and $\dim(L) = \dim(\widetilde{L}) = m$, there exists an integer $(n-m) \times n$-matrix A such that $\widetilde{L} = \operatorname{Ker}_{\mathbb{Z}}(A)$ (see [6, Proposition 2.1]). Moreover, by the elementary divisor theorem [1, Theorem 2.4.13], there exist positive integers $d_1, \dots, d_m$ called the elementary divisors of L in $\widetilde{L}$ such that d_{i+1} divides d_i for $i = 1, \dots, m-1$, and $\widetilde{L}/L \simeq \bigoplus_{i=1}^m \mathbb{Z}/d_i\mathbb{Z}$. Hence $\widetilde{L}/L$ is a finite group of order $d_1 \cdots d_m$. Furthermore, there exists a basis $\mathbf{b}_1, \dots, \mathbf{b}_m$ for $\widetilde{L}$ as a lattice, such that $d_1\mathbf{b}_1, \dots, d_m\mathbf{b}_m$ is a basis of L.

We can now formulate the main result of the paper.

2. The main result

We use the same definitions and notation as in Section 1.

Theorem 2.1
Let L be a lattice in $\mathbb{Z}^n$ of dimension m, $\widetilde{L}$ be its saturation, and let B be a defining matrix of L. With the notation as above, the following conditions hold true.

(i) *If B' is another defining matrix of L, then*

$$\mathbf{C}^B_{k_1,\dots,k_{m-1}} = \pm\mathbf{C}^{B'}_{k_1,\dots,k_{m-1}} \quad \text{and} \quad \beta^B_{k_1,\dots,k_{m-1}} = \beta^{B'}_{k_1,\dots,k_{m-1}}.$$

(ii) *The circuits of $\widetilde{L}$ are precisely of the form $\pm\bar{\mathbf{C}}^B_{k_1,\dots,k_{m-1}}$, where*

$$\bar{\mathbf{C}}^B_{k_1,\dots,k_{m-1}} := \mathbf{C}^B_{k_1,\dots,k_{m-1}} / \beta^B_{k_1,\dots,k_{m-1}}.$$

(*iii*) *The circuits of* L *are precisely of the form*

$$\pm|\bar{\mathbf{C}}^B_{k_1,\dots,k_{m-1}}|\,\bar{\mathbf{C}}^B_{k_1,\dots,k_{m-1}},$$

where $|\bar{\mathbf{C}}^B_{k_1,\dots,k_{m-1}}|$ *denotes the order, as a group element, of the image of* $\bar{\mathbf{C}}^B_{k_1,\dots,k_{m-1}}$ *in* $\widetilde{L}/L$ *under the natural surjection* $\widetilde{L}\to\widetilde{L}/L$.

A version of Theorem 2.1 for the circuits of linear spaces is given (without any proof) in [9, p. 3]. It is also restated in [2, Lemma 4.1.4], and is proved using properties of Gröbner bases. Here, we prove Theorem 2.1 by the properties of integer lattices.

Proof of Theorem 2.1 (*i*). Since $B' = BU$ for a unimodular matrix U, then for any sequence $1\le i_1,\dots,i_m\le n$, we have

$$B'[i_1,\dots,i_m|1,\dots,m] = B[i_1,\dots,i_m|1,\dots,m]U.$$

Therefore, depending on the value of $\det U$ which is $+1$ or -1 because U is unimodular, we have $D_{B'}[i_1,\dots,i_m] = \pm D_B[i_1,\dots,i_m]$. This implies that $\mathbf{C}^B_{k_1,\dots,k_{m-1}} = \pm\,\mathbf{C}^{B'}_{k_1,\dots,k_{m-1}}$, and hence $\beta^B_{k_1,\dots,k_{m-1}} = \beta^{B'}_{k_1,\dots,k_{m-1}}$. Note that the greatest common divisor is assumed to be positive.

Let $\widetilde{B}$ be a defining matrix of $\widetilde{L}$. In view of Theorem 2.1 (*i*), and the elementary divisor theorem [1, Theorem 2.4.13], we may assume that $\mathbf{b}_1,\dots,\mathbf{b}_m$ are the columns of $\widetilde{B}$, and there exist positive integers $d_1,\dots,d_m$ such that $d_1\mathbf{b}_1,\dots,d_m\mathbf{b}_m$ are the columns of B. It is clear that

$$D_B[i_1,\dots,i_m] = d_1\cdots d_m D_{\widetilde{B}}[i_1,\dots,i_m].$$

To prove Theorem 2.1 (*ii*), we need some elementary lemmas concerning the lattice $\widetilde{L}$. We assume that $\mathbf{a}_1,\dots,\mathbf{a}_n$ are the columns of A, where A is an integer $(n-m)\times n$-matrix with the property that $\widetilde{L} = \mathrm{Ker}_{\mathbb{Z}}(A)$. We also define that $\mathrm{codim}(L) := d := n-m$, and hence $\mathrm{rank}(A) = d$.

Lemma 2.2
Let $\mathbf{u}\in L$ *be a circuit with* $\mathrm{supp}(\mathbf{u}) = \{i_1,\dots,i_t\}$ *and let* A' *be the submatrix of* A *whose columns are* $\mathbf{a}_{i_1},\dots,\mathbf{a}_{i_t}$. *Then* $\mathrm{rank}(A') = t-1$.

Proof. By [6, Proposition 2.4], $\widetilde{L}$ has a circuit $\widetilde{\mathbf{u}}$ with the same support, that is, $\mathbf{u} = \alpha\cdot\widetilde{\mathbf{u}}$ for some $\alpha\in\mathbb{Z}$. Let s be the number of independent columns of the matrix A'. Then clearly $s\le t$. Without loss of generality, we may assume that $\mathbf{a}_{i_1},\dots,\mathbf{a}_{i_s}$ are independent columns of A'. If $s=t$, then the columns of A' are independent, and hence $\mathrm{supp}(\widetilde{\mathbf{u}}) = \emptyset$ which is impossible. If $s\le t-2$, then the columns $\mathbf{a}_{i_1},\dots,\mathbf{a}_{i_s},\mathbf{a}_{i_{s+1}}$ are linearly dependent, and hence there exist relatively prime integers $\lambda_1,\dots,\lambda_{s+1}$ which are not simultaneously zero and they satisfy $\lambda_1\mathbf{a}_{i_1}+\cdots+\lambda_s\mathbf{a}_{i_s}+\lambda_{s+1}\mathbf{a}_{i_{s+1}} = 0$. Then for the vector $\mathbf{v} = \lambda_1\mathbf{e}_{i_1}+\cdots+\lambda_s\mathbf{e}_{i_s}+\lambda_{s+1}\mathbf{e}_{i_{s+1}}$ in which $\mathbf{e}_{i_1},\dots,\mathbf{e}_{i_s},\mathbf{e}_{i_{s+1}}$ are standard vectors of $\mathbb{Z}^n$, we have $\mathrm{supp}(\mathbf{v})\subseteq\{i_1,\dots,i_{s+1}\}$. Since $s+1\le t-1$, we have $\mathrm{supp}(\mathbf{v})\subsetneqq\mathrm{supp}(\widetilde{\mathbf{u}})$ contradicting $\widetilde{\mathbf{u}}$ is a circuit of $\widetilde{L}$. Hence the only possible case is $s=t-1$.

Corollary 2.3 (cf. [9, Lemma 4.8])
If $\mathbf{u} \in L$ *is a circuit, then* $\operatorname{supp}(\mathbf{u})$ *has at most* $\operatorname{codim}(L)+1$ *elements. In other words,* $|\operatorname{supp}(\mathbf{u})| \leq d+1$*.*

Proof. Follows directly from Lemma 2.2.

Lemma 2.4 (cf. [9, Lemma 4.9] and [10, Proposition 1.9.9])
The circuits of $\widetilde{L}$ *are precisely all vectors of the form*

$$\mathbf{u} = \pm \frac{1}{\alpha_{j_1 \cdots j_{d+1}}} \sum_{i=1}^{d+1} (-1)^i \det(\mathbf{a}_{j_1}, \ldots, \widehat{\mathbf{a}_{j_i}}, \ldots, \mathbf{a}_{j_{d+1}}) \mathbf{e}_{j_i}$$

where $\{j_1, \ldots, j_{d+1}\} \subseteq \{1, \ldots, n\}$*, and* $\alpha_{j_1 \cdots j_{d+1}}$ *is the greatest common divisor of* $\det(\mathbf{a}_{j_1}, \ldots, \widehat{\mathbf{a}_{j_i}}, \ldots, \mathbf{a}_{j_{d+1}})$ *for* $i = 1, \ldots, d+1$*, and it is assumed to be positive.*

Proof. We prove the result in two steps.

1st Step. Let $n = d+1$. Then $j_i = i$ for $i = 1, \ldots, d+1$, and $\dim(\widetilde{L}) = 1$. Since $\operatorname{rank}(A) = d$, there is some i for which the columns $\mathbf{a}_1, \ldots, \widehat{\mathbf{a}}_i, \ldots, \mathbf{a}_{d+1}$ of the matrix A are linearly independent. For a nonzero element $\mathbf{u} = (u_1, \ldots, u_{d+1}) \in \widetilde{L}$, we have

$$u_1 \mathbf{a}_1 + \cdots + u_{i-1}\mathbf{a}_{i-1} + u_{i+1}\mathbf{a}_{i+1} \cdots + u_{d+1}\mathbf{a}_{d+1} = -u_i \mathbf{a}_i.$$

Clearly, $u_i \neq 0$, and using Cramer's rule of elementary linear algebra, we have

$$\begin{aligned} u_j &= \frac{\det(\mathbf{a}_1, \ldots, \widehat{\mathbf{a}}_i, \ldots, -u_i\mathbf{a}_i, \ldots, \mathbf{a}_{d+1})}{\det(\mathbf{a}_1, \ldots, \widehat{\mathbf{a}}_i, \ldots, \mathbf{a}_{d+1})} \\ &= (-1)^{i-j} u_i \frac{\det(\mathbf{a}_1, \ldots, \widehat{\mathbf{a}}_j, \ldots, \mathbf{a}_{d+1})}{\det(\mathbf{a}_1, \ldots, \widehat{\mathbf{a}}_i, \ldots, \mathbf{a}_{d+1})} \end{aligned}$$

for $j = 1, \ldots, d+1$. Substituting the above values of u_j in the expression $\mathbf{u} = \sum_{j=1}^{d+1} u_j \mathbf{e}_j$, we see that $\mathbf{u} = \frac{\beta}{\alpha} \sum_{j=1}^{d+1} (-1)^j \det(\mathbf{a}_1, \ldots, \widehat{\mathbf{a}}_j, \ldots, \mathbf{a}_{d+1}) \mathbf{e}_j$, where α, β are integers and are assumed to be relatively prime. If, in addition, we assume that $\mathbf{u}$ is a circuit of $\widetilde{L}$, then it is easy to show that $\beta = \pm 1$, and α is the desired greatest common divisor.

2nd Step. First we assume that $\mathbf{u} \in \widetilde{L}$ is a circuit with $\operatorname{supp}(\mathbf{u}) = \{i_1, \ldots, i_t\}$. Then $t \leq d+1$, by Corollary 2.3. The submatrix A' whose columns are $\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_t}$ has rank $t-1$, by Lemma 2.2. Since $\operatorname{rank}(A) = d$, we can add $d-t+1$ columns of A to the matrix A' such that the resulting $d \times (d+1)$-matrix A'' has rank d. Let $\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_{d+1}}$ be the columns of A''. Note that $\{i_1, \ldots, i_t\} \subseteq \{j_1, \ldots, j_{d+1}\}$. The restriction $\bar{\mathbf{u}}$ of $\mathbf{u}$ to $\{j_1, \ldots, j_{d+1}\}$ lives in $\operatorname{Ker}_{\mathbb{Z}}(A'')$, and hence it has the desired form, by the first step. This shows that $\mathbf{u}$ is also of desired form.

Conversely, assume that $\mathbf{u}$ has the form given by the lemma. We want to show that $\mathbf{u}$ is a circuit of $\widetilde{L}$. Since $\mathbf{u}$ is nonzero, at least one of the determinants is nonzero. This implies that the matrix with the columns $\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_{d+1}}$ has rank d. Hence $\mathbf{u} \in \widetilde{L} = \operatorname{Ker}_{\mathbb{Z}}(A)$, by the first step. Since the coordinates of $\mathbf{u}$ are relatively prime, we only have to show that the support of $\mathbf{u}$ is minimal.

Let $\mathbf{v} \in \widetilde{L}$, and $\mathrm{supp}(\mathbf{v}) \subseteq \mathrm{supp}(\mathbf{u})$. For an element $j_i \in \mathrm{supp}(\mathbf{u})$, we have $\det(\mathbf{a}_{j_1}, \ldots, \widehat{\mathbf{a}_{j_i}}, \ldots, \mathbf{a}_{j_{d+1}}) \neq 0$, by definition of $\mathbf{u}$. This shows that

$$\{\mathbf{a}_{j_1}, \ldots, \widehat{\mathbf{a}_{j_i}}, \ldots, \mathbf{a}_{j_{d+1}}\}$$

is linearly independent, and hence $j_i \in \mathrm{supp}(\mathbf{v})$. Therefore $\mathrm{supp}(\mathbf{v}) = \mathrm{supp}(\mathbf{u})$, as required.

Proof of Theorem 2.1 (*ii*). Let $\mathbf{u}$ be a circuit in $\widetilde{L}$. Let moreover

$$\sigma := \{1, \ldots, n\} \setminus \{j_1, \ldots, j_{d+1}\} \quad \text{and} \quad \sigma_i := \sigma \cup \{j_i\}.$$

Note that each subset of $\{1, \ldots, n\}$ is ordered by the ordering induced from $1 < 2 < \cdots < n$. Using Lemma 2.4, and [6, Theorem 2.8], we can write

$$\mathbf{u} = \pm \frac{1}{\alpha_{j_1 \cdots j_{d+1}}} \sum_{i=1}^{d+1} (-1)^i (-1)^{1+\cdots+d+\sum_{i \in \sigma_i} i} C(A) D_{\widetilde{B}}[\sigma_i] \mathbf{e}_{j_i}.$$

Here $C(A)$ stands for the greatest common divisor of all maximal minors of the matrix A. We may assume that

$$\alpha_{j_1 \cdots j_{d+1}} = C(A) \alpha'_{j_1 \cdots j_{d+1}}$$

for some positive integer $\alpha'_{j_1 \cdots j_{d+1}}$. If we move the row of $\widetilde{B}[\sigma_i | 1, \ldots, m]$ corresponding to the index j_i to the bottom of the matrix, then it is easy to see that

$$D_{\widetilde{B}}[\sigma_i] = (-1)^{n - j_i - (d+1-i)} D_{\widetilde{B}}[(\sigma, j_i)],$$

where (σ, j_i) is the ordered set obtained from σ by adding j_i to it as the last element. Therefore

$$\begin{aligned} \mathbf{u} &= \pm \frac{1}{\alpha'_{j_1 \cdots j_{d+1}}} \sum_{i=1}^{d+1} (-1)^{2(i-j_i)} (-1)^{1+\cdots+d+\sum_{i \in \sigma} i} (-1)^{m-1} D_{\widetilde{B}}[(\sigma, j_i)] \mathbf{e}_{j_i} \\ &= \pm \frac{1}{\alpha'_{j_1 \cdots j_{d+1}}} \sum_{i=1}^{d+1} D_{\widetilde{B}}[(\sigma, j_i)] \mathbf{e}_{j_i}. \end{aligned}$$

Let $\sigma := \{k_1, \ldots, k_{m-1}\}$, then

$$\begin{aligned} \mathbf{u} &= \pm \frac{1}{\alpha'_{j_1 \cdots j_{d+1}}} \sum_{i=1}^{d+1} D_{\widetilde{B}}[k_1, \ldots, k_{m-1}, j_i] \mathbf{e}_{j_i} \\ &= \pm \frac{1}{d_1 \cdots d_m \alpha'_{j_1 \cdots j_{d+1}}} \sum_{i=1}^{d+1} D_B[k_1, \ldots, k_{m-1}, j_i] \mathbf{e}_{j_i}. \end{aligned}$$

If $i = k_\ell$ for some $\ell \in \{1, \ldots, m-1\}$, then $D_{\widetilde{B}}[k_1, \ldots, k_{m-1}, i] = 0$. Thus

$$\begin{aligned}\mathbf{u} &= \pm\frac{1}{d_1\cdots d_m\alpha'_{j_1\cdots j_{d+1}}}\sum_{i=1}^{d+1}D_B[k_1,\ldots,k_{m-1},i]\mathbf{e}_i\\ &= \pm\frac{1}{\beta^B_{k_1,\ldots,k_{m-1}}}\sum_{i=1}^{d+1}D_B[k_1,\ldots,k_{m-1},i]\mathbf{e}_i = \pm\bar{\mathbf{C}}^B_{k_1,\ldots,k_{m-1}},\end{aligned}$$

as desired.

Proof of Theorem 2.1 (iii). By [6, Proposition 2.4], there is a one to one correspondence between the circuits of L and those of $\widetilde{L}$. In fact, $\mathbf{u}\in\widetilde{L}$ is a circuit of $\widetilde{L}$ if and only if $m\mathbf{u}$ is a circuit of L, where m is the smallest positive integer with the property that $m\mathbf{u}\in L$. But such an integer m is clearly the order of $\mathbf{u}+L$ as a group element of $\widetilde{L}/L$. Therefore, the result follows from (ii) of Theorem 2.1.

Example 2.5
Let L be a lattice in $\mathbb{Z}^3$ whose basis is $\{(6,-2,4),(2,2,0)\}$. Using Theorem 2.1, we compute the circuits of L. It is appropriate to use a computer algebra system for computations, but in this example all computations can be easily checked by hand. Clearly $\{(8,0,4),(2,2,0)\}$ is another basis of L. It follows from the shape of this basis that $\{(2,0,1),(1,1,0)\}$ is a basis of $\widetilde{L}$, and $d_1=4, d_2=2$ are the elementary divisors of L in $\widetilde{L}$. The matrix

$$B=\begin{bmatrix}8&2\\0&2\\4&0\end{bmatrix}$$

is a defining matrix of L. We have $n=3, m=2$, and hence 6 circuits by Theorem 2.1 which are $\pm\bar{\mathbf{C}}^B_{k_1}$ for $k_1=1,2,3$. More precisely, we have

$$\mathbf{C}^B_{k_1}=D_B[k_1,1]\mathbf{e}_1+D_B[k_1,2]\mathbf{e}_2+D_B[k_1,3]\mathbf{e}_3.$$

Therefore,

1) $\mathbf{C}^B_1=D_B[1,2]\mathbf{e}_2+D_B[1,3]\mathbf{e}_3=16\mathbf{e}_2-8\mathbf{e}_3$, $\beta^B_1=8$, and $\bar{\mathbf{C}}^B_1=2\mathbf{e}_2-\mathbf{e}_3$.
2) $\mathbf{C}^B_2=D_B[2,1]\mathbf{e}_1+D_B[2,3]\mathbf{e}_3=-16\mathbf{e}_1-8\mathbf{e}_3$, $\beta^B_2=8$, and $\bar{\mathbf{C}}^B_2=-2\mathbf{e}_1-\mathbf{e}_3$.
3) $\mathbf{C}^B_3=D_B[3,1]\mathbf{e}_1+D_B[3,2]\mathbf{e}_2=8\mathbf{e}_1+8\mathbf{e}_2$, $\beta^B_3=8$, and $\bar{\mathbf{C}}^B_3=\mathbf{e}_1+\mathbf{e}_2$.

It is easy to see that $|\bar{\mathbf{C}}^B_1|=|\bar{\mathbf{C}}^B_2|=4$, and $|\bar{\mathbf{C}}^B_3|=2$. Thus the circuits of L are

1) $\pm|\bar{\mathbf{C}}^B_1|\bar{\mathbf{C}}^B_1=\pm4(2\mathbf{e}_2-\mathbf{e}_3)=\pm(0,8,-4)$.
2) $\pm|\bar{\mathbf{C}}^B_2|\bar{\mathbf{C}}^B_2=\pm4(-2\mathbf{e}_1-\mathbf{e}_3)=\pm(-8,0,-4)$.
3) $\pm|\bar{\mathbf{C}}^B_3|\bar{\mathbf{C}}^B_3=\pm2(\mathbf{e}_1+\mathbf{e}_2)=\pm(2,2,0)$.

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