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Report of Meeting

19th International Conference on Functional Equations and Inequalities, Będlewo, Poland, September 11–18, 2021

The **19th International Conference on Functional Equations and Inequalities** (19th ICFEI) took place in the *Mathematical Research and Conference Center* in Będlewo (Poland) on September 12–18, 2021. It was organized by the *Department of Mathematics of the Pedagogical University of Krakow* and the *Stefan Banach International Mathematical Center*.

The Scientific Committee of the 19th ICFEI consisted of Professors: Nicole Brillouët-Belluot (France), Janusz Brzdęk (Poland) – chairman, Jacek Chmieliński (Poland), Roman Ger (Poland), Zsolt Páles (Hungary), Dorian Popa (Romania), Ekaterina Shulman (Poland/Russia), Henrik Stetkær (Denmark), László Székelyhidi (Hungary) and Marek Cezary Zdun (Poland). The Organizing Committee consisted of Jacek Chmieliński (chairman), Eliza Jabłońska (vice chairman), Beata Deręgowska (managing secretary), Zbigniew Leśniak (secretary), Paweł Pasteczka (secretary), Paweł Wójcik (scientific secretary) and Paweł Solarz (web & technical support).

Because of the Covid-19 pandemic time, the conference was organized on a hybrid basis. There were 112 registered participants (not counting a certain number of guests attending particular sessions), including 45 present in Będlewo. All together 22 countries were represented: Poland (42 participants), Hungary (17), China (10), Romania (10), India (8), Morocco (3), Austria (2), Egypt (2), Germany (2), Iran (2), Italy (2), Japan (2) and – with one participant – Chile, Croatia, Czech Republic, Namibia, Republic of Korea, South Africa, Spain, Taiwan, United Kingdom and United States. Participants staying in Będlewo represented 9 countries: Poland (27), Hungary (7), Romania (5) and – with one participant – Austria, Germany, Italy, Czech Republic, Namibia and United States.

The conference was officially opened (remotely from Kraków) on Monday, September 13, by Professor Tomasz Szemberg – the Head of the Department of Mathematics of the Pedagogical University of Krakow. After that a welcome address and general announcements were given by Professor Jacek Chmieliński, the Chairman of the Organizing Commiteee.

During 21 scientific sessions, 75 talks were presented. Four of them were longer invited lectures and were delivered by Professors Attila Gilanyi, Weinian Zhang, Soon-Mo Jung and Adrian Petrusel. The talks were devoted mainly to functional equations and inequalities, their stability, convexity, means, as well as to related topics in real analysis, functional analysis, applications of mathematics and others. Additionally, apart from regular talks, spontaneous contributions in *Problems and Remarks* sessions were delivered.

Some social events accompanied the conference. A picnic with a bonfire was organized on Tuesday evening and a banquet on Wednesday. On Thursday evening a guitar and piano concerts were performed by Professors Grzegorz Guzik and László Székelyhidi.

The conference was closed on Saturday, September 18, by Professor Roman Ger on behalf of the Scientific Commitee. The subsequent 20th ICFEI has been announced to be organized on September 17–23, 2023, again in Będlewo.

1. Abstracts of Talks

🌍 – online talks

Ana Maria Acu 🌍 *Applications of Ohlin lemma and Levin-Stečkin inequalities* (joint work with Ioan Raşa)

Let $n \geq 1$ and $a_{n,j} := 4^{-n}\binom{2j}{j}\binom{2n-2j}{n-j}$, $j = 0, 1, \ldots, n$. The probability distribution $(a_{n,j})_{j=0,\ldots,n}$ is involved in the study of some entropies: see [1], where two proofs of the inequality

$$\frac{1}{n+1}\sum_{j=0}^{n} f(j) \leq \sum_{j=0}^{n} a_{n,j} f(j), \qquad f\colon \mathbb{R} \to \mathbb{R} \text{ convex}, \tag{1}$$

are given. One proof is based on the Ohlin Lemma. In this talk we discuss the connection of (1) with the Levin-Stečkin inequalities. New inequalities are obtained for strongly convex functions, combining (1) with results from [2]. Finally, we obtain estimates for the information potential associated with the distribution $(a_{n,j})_{j=0,\ldots,n}$.

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Mirosław Adamek *Some remarks about convex sequences*

Let Z be a set of consecutive integers with $\operatorname{card}(Z) \geq 3$. A sequence of real numbers $(a_n)_{n \in Z}$ is called convex (see [2]), if it satisfies the following inequality

$$a_n \leq \frac{a_{n-1} + a_{n+1}}{2}$$

for all $n-1, n, n+1 \in Z$.

In this talk we will recall some known facts about convex sequences and give a discreet version of the result presented in [1], i.e. we will give a necessary and sufficient condition for the separation of two sequences with a convex sequence. As a consequence of a separation theorem we will obtain Hyers-Ulam stability type result for convex sequences. We will also give, as a consequence of the presented results for convex sequences, results concerning generalized convex sequences in the sense of [3].

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Javid Ali 🌍 *Fixed Point Iterations: Convergence, stability and data dependence results*

In this talk we discuss a newly introduced two step fixed point iterative algorithm. We prove, under certain conditions, a strong convergence result for weak contractions $T\colon C \to C$, where C is a nonempty, closed and convex subset of a Banach space. We also prove stability and data dependency of a proposed iterative algorithm. Furthermore, we utilize our main result to approximate the solution of a nonlinear functional Volterra integral equation.

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Ljiljana Arambašić 🌍 *Strong Birkhoff-James orthogonality in commutative C^*-algebras* (joint work with Alexander Guterman, Bojan Kuzma, Rajna Rajić, and Svetlana Zhilina)

General normed spaces are not equipped with an inner product. Nonetheless, there do exist several nonequivalent extensions of orthogonality from inner product spaces to general normed ones. One of the most well-known is the Birkhoff-James orthogonality: if X is a normed space and $x, y \in X$, then x is Birkhoff-James orthogonal to y if $\|x+\lambda y\| \geq \|x\|$ holds for all scalars λ. In a C^*-algebra A we can also discuss the case when $x \in A$ is Birkhoff-James orthogonal to all the elements of the form $ya, a \in A$. In this case we say that x is strongly Birkhoff-James orthogonal to y. We discuss this kind of orthogonality in commutative C^*-algebras.

Karol Baron *Strong law of large numbers for iterates of some random-valued functions* (joint work with Rafał Kapica)

Fix a probability space $(\Omega, \mathcal{A}, P)$ and a metric space X. Let $\mathcal{B}$ denote the σ-algebra of all Borel subsets of X. We say that $f\colon X \times \Omega \to X$ is a *random-valued* function (an *rv-function* for short) if it is measurable with respect to the product σ-algebra $\mathcal{B} \otimes \mathcal{A}$. The iterates of such an rv-function are given by

$$f^0(x, \omega_1, \omega_2, \ldots) = x, \quad f^n(x, \omega_1, \omega_2, \ldots) = f\big(f^{n-1}(x, \omega_1, \omega_2, \ldots), \omega_n\big)$$

for $n \in \mathbb{N}$, $x \in X$ and $(\omega_1, \omega_2, \ldots)$ from Ω^∞ defined as $\Omega^{\mathbb{N}}$. Note that $f^n\colon X \times \Omega^\infty \to X$ is for every $n \in \mathbb{N}$ an rv-function on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$. See [4], Sec. 1.4, and [2].

Many results on the convergence of such iteration sequences can be found in the literature. They concern different type of convergence and are given under suitable assumptions concerning both the metric space and the rv-function. We accept the following hypothesis:

(H) (X, ρ) is a complete and separable metric space and $f\colon X \times \Omega \to X$ is an rv-function such that

$$\int_\Omega \rho\big(f(x,\omega), f(z,\omega)\big) P(d\omega) \leq \lambda \rho(x,z) \qquad \text{for } x, z \in X$$

with a $\lambda \in (0,1)$, and

$$\int_\Omega \rho\big(f(x,\omega), x\big) P(d\omega) < \infty \qquad \text{for } x \in X.$$

Then (see [1] and [3]) there exists a probability Borel measure π^f on X such that for every $x \in X$ the sequence of distributions of $f^n(x, \cdot)$, $n \in \mathbb{N}$, converges weakly to π^f and

$$\int_X \rho(x,z) \pi^f(dz) < \infty.$$

We are looking for conditions that imposed on the rv-function $f\colon X \times \Omega \to X$ and a function $\psi\colon X \to \mathbb{R}$ provide that for every $x \in X$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \psi \circ f^k(x, \cdot) = \int_X \psi d\pi^f \qquad \text{a.e. for } P^\infty.$$

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Chaimaa Benzarouala 🌍 *A fixed point approach to the Ulam-Hyers-Rassias stability of a generalized linear functional equation* (joint work with Lahbib Oubbi)

During this talk, we introduce the general functional equation

$$\sum_{i=1}^{m} A_i\big(f(\varphi_i(\bar{x}))\big) + b = 0, \qquad \bar{x} := (x_1, \ldots, x_n) \in X^n$$

and study its Ulam-Hyers-Rassias stability and hyperstability, using a fixed point approach, where m and n are positive integers, f is a mapping from a vector space X into a Banach space $(Y, \|\cdot\|)$, and for every $i \in \{1, \ldots, m\}$, φ_i is a linear mapping from X^n into X, A_i is a continuous endomorphism of Y and $b \in Y$. Our result covers most of the former ones in the literature concerning the stability and hyperstability of linear functional equations, as well as new situations. This talk is based on the article [3].

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Zoltán Boros *Conditionally linked monomial functions* (joint work with Edit Garda-Mátyás)

Among others, we establish the following generalized versions of our recent statements [2].

Theorem 1
Suppose that $f, g\colon \mathbb{R} \to \mathbb{R}$ are generalized monomials of degree $n \in \mathbb{N}$ that satisfy the additional equation $y^n f(x) = x^n g(y)$ under the condition $y = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$, where $m \in \mathbb{N}$, $a_i \in \mathbb{R}$ $(i = 0, \ldots, m)$ are fixed such that $a_m \neq 0$ and $a_0 \neq 0$. Then $f(x) = g(x) = x^n f(1)$ for all $x \in \mathbb{R}$.

Theorem 2
Let $n \in \{1, 2, 3\}$. If $m \in \mathbb{Z}$, $|m| \geqslant 2$ and $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized monomial of degree n that satisfies the additional equation $y^n f(x) = x^n f(y)$ under the condition $y = x^m$, then $f(x) = x^n f(1)$ for all $x \in \mathbb{R}$.

Our former results [1] for the case $n = 2$ indicate that we cannot immediately extend the latter statement for two generalized monomials in the conditional equation.

Further similar problems, statements and counterexamples are presented as well.

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Pál Burai *Random means generated by random variables: expectation and limit theorems* (joint work with Mátyás Barczy)

We introduce the notion of a random mean generated by a random variable and give a construction of its expected value. We derive some sufficient conditions under which strong law of large numbers and some limit theorems hold for random means generated by the elements of a sequence of independent and identically distributed random variables.

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Gopalakrishna Chaitanya *Square roots of functions*

An *iterative square root* of a function f is a function g such that

$$g\big(g(\cdot)\big) = f(\cdot).$$

We discuss a new result on the nonexistence of iterative square roots of self-maps on arbitrary sets and use it to show that continuous self-maps without iterative square roots are dense in the space of all continuous self-maps on the unit cube in $\mathbb{R}^m$. This talk is based on a joint work with B. V. Rajarama Bhat. (see https://arxiv.org/abs/2105.02171).

Jacek Chmieliński *On continuity of seminorms* (joint work with Moshe Goldberg)

Let X be an infinite-dimensional real or complex vector space. We study the continuity and discontinuity properties of seminorms with respect to norm-topologies on X. In particular, the following problems are considered:

- the existence of a norm on X, with respect to which a given seminorm is continuous/discontinuous;
- the existence of a norm on X, with respect to which all seminorms are continuous/discontinuous;
- the equivalence of norms, such that a given seminorm is continuous/discontinuous with respect to all of them.

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Jacek Chudziak *On existence and properties of the principle of equivalent utility*

The principle of equivalent utility is one of the methods of insurance contract pricing. If $w \in \mathbb{R}$ is an initial wealth of the insurance company and $u\colon \mathbb{R} \to \mathbb{R}$ is its continuous and strictly increasing utility function, then the premium of equivalent utility for a risk X, represented by a non-negative bounded random variable on a given probability space, is defined as a solution $H_{(w,u)}(X)$ of the equation

$$E[u(w + H_{(w,u)}(X) - X)] = u(w). \tag{1}$$

Several results concerning the principle defined by (1) can be found e.g. in Bowers et al. [1], Bühlmann [2], Gerber [3] and Rolski et al. [6].

Recently, the principle of equivalent utility has been extended onto various models of decision making under risk. Heilpern [4] proposed and investigated the principle under rank-dependent utility model. Kałuszka and Krzeszowiec [5] introduced the principle of equivalent utility based on Cumulative Prospect Theory. In this setting the principle is defined through the equation

$$E_{gh}\big[u\big(w + H_{(w,u,g,h)}(X) - X\big)\big] = u(w), \tag{2}$$

where E_{gh} is the the generalized Choquet integral with respect to to the probability weighting functions g (for gains) and h (for losses) – cf. [5].

In the talk we establish a necessary and sufficient condition for the existence and uniqueness of the principle defined by (2). Furthermore we provide the characterizations of some important properties of the principlpe.

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Beata Deręgowska *Quasiarithmetic-type invariant means on a probability space* (joint work with Paweł Pasteczka)

For a family $(\mathscr{A}_x)_{x\in(0,1)}$ of integral quasi-arithmetic means satisfying certain measurability-type assumptions we search for an integral mean K such that

$$K\big((\mathscr{A}_x(\mathbb{P}))_{x\in(0,1)}\big) = K(\mathbb{P})$$

for every compactly supported probability Borel measure $\mathbb{P}$. Also some results concerning the uniqueness of invariant means will be given.

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Hojjat Farzadfard 🌍 *On the real Poincaré functional equation*

Let I be a real interval, $f\colon I \to I$ and $\alpha \in \mathbb{R}$. A real Poincaré functional equation is of the form

$$\psi(\alpha x) = f\big(\psi(x)\big), \qquad x \in I,$$

where a function $\psi\colon \mathbb{R} \to I$ is unknown. We deal with the case where I is a compact interval and a continuous periodic solution ψ is desired. This talk is based on the papers [1] and [2]. The main results are as follows:

Definition 1
We call a function $\psi\colon \mathbb{R} \to \mathbb{R}$ a *cosine-like function* provided that ψ is continuous, nonconstant and periodic, moreover, if λ is the principal period of ψ, then

1. ψ is even, in particular $\psi(x) = \psi(\lambda - x)$ for all $0 \le x \le \lambda$.
2. If x and y are two distinct points in $[0, \lambda]$ and $\psi(x) = \psi(y)$, then $y = \lambda - x$.
3. ψ attains one of its absolute extrema at 0 and the other at $\lambda/2$.
4. ψ is strictly increasing on one of the intervals $[0, \lambda/2]$ and $[\lambda/2, \lambda]$ and strictly decreasing on the other.

Definition 2
Let I be a compact interval of positive length, $f\colon I \to I$ and $k \in \mathbb{N}$. We say that f is a *fluctuating function of rank* $\boldsymbol{k}$ provided that there exists a partition $a_0 < a_1 < \ldots < a_k$ of I such that for each $0 \le j < k$ the function f is strictly monotone on the interval $[a_j, a_{j+1}]$ and $f\big([a_j, a_{j+1}]\big) = I$. The *node vector* of f is denoted by $\mathrm{Node}(f) = (a_0, \ldots, a_k)$. Also

$$\mathrm{Node}_\infty(f) := \bigcup_{n=1}^{\infty} \mathrm{Node}(f^n).$$

Theorem 1 (The Existence Theorem)
Let $I = [a, b]$ be a compact real interval and $f : I \to I$ be a fluctuating function of rank $k > 1$. The following statements are equivalent:

1. *For every $\lambda > 0$ the equation $\psi(kx) = f\big(\psi(x)\big)$ has a solution ψ which is a cosine-like function with the principal period λ.*
2. $\mathrm{Node}_\infty(f)$ *is dense in I.*

Theorem 2 (The Uniqueness Theorem)
Let $\psi : \mathbb{R} \to I$ and $\phi : \mathbb{R} \to I$ be two solutions of the Poincaré functional equation $\psi(kx) = f(\psi(x))$ with ψ cosine-like and ϕ non-constant, continuous and periodic, where $k > 1$ is an integer. Then there exist a positive real number ω and an integer m such that $\phi(x) = \psi(\omega x + m\lambda/(k-1))$, $(x \in \mathbb{R})$, where λ is the principal period of ψ.

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Żywilla Fechner *Moment functions and exponential monomials on commutative hypergroups* (joint work with Eszter Gselmann and László Székelyhidi)

In this talk we are going to discuss a connection between moment functions and exponential monomials on commutative hypergroups. At the beginning we are going to give a brief motivation of the problem. In particular, we formalize the notions of an exponential monomial and moment functions. We want to find conditions under which exponential monomials can be expressed in terms of generalized moment functions. We present one of possible conditions and discuss some possible ways for further researchs.

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Gian Luigi Forti *Alternative Cauchy equation in three unknown functions*

In this paper we deal with the product of two or three Cauchy differences equaled to zero. We show that in the case of two Cauchy differences, the condition of absolute continuity and differentiability of the two functions involved implies that one of them must be linear, i.e. we have a trivial solution. In the case of the product of three Cauchy differences the situation changes drastically: there exists non trivial $\mathcal{C}^\infty$ solutions, while in the case of real analytic functions we obtain that at least one of the functions involved must be linear. Some open problems are presented.

Roman Ger *On alienation of two functional equations of quadratic type (again)*

We deal with an alienation problem for an Euler-Lagrange type functional equation

$$f(\alpha x+\beta y)+f(\alpha x-\beta y)=2\alpha^2 f(x)+2\beta^2 f(y)$$

assumed for fixed nonzero real numbers α, β, $1 \neq \alpha^2 \neq \beta^2$, and the classic quadratic functional equation

$$g(x+y)+g(x-y)=2g(x)+2g(y).$$

We were inspired by papers of Chang Il Kim, Giljun Han & Seong-A. Shim [3] and M. Eshagi Gordji & H. Khodaei [2], where the special case $g=\gamma f$ was examined, as well as by a suggestion of Bessenyei Mihály [1].

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Attila Gilányi *Computer assisted studies of functional equations and inequalities*

During the last 30 years, computer assisted methods were applied in investigations of functional and inequalities by several researchers. Among others, solutions of general two variable linear functional equations were determined and further studies of linear equations were performed by computer, regularity properties of certain functional equations were tested, furthermore, various classes of means were studied with the help of such methods. In this talk, we present a systematic survey of these results.

Dorota Głazowska *Generalized classical weighted means and some related problems* (joint work with Janusz Matkowski)

Under some simple conditions on real function f defined on an interval I, the two-variable functions given by the following formulas

$$A_f(x,y) := f(x)+y-f(y), \quad G_f(x,y) := \frac{f(x)}{f(y)}\, y$$

and

$$H_f(x,y) := \frac{xy}{f(x)+y-f(y)}$$

for all $x, y \in I$, generalize, respectively, the classical weighted arithmetic, geometric and harmonic means. In fact these means are symmetric, if and only if they coincide with $\mathcal{A}$, $\mathcal{G}$, $\mathcal{H}$, respectively. The invariance identity

$$\mathcal{G} \circ (A_f, H_f) = \mathcal{G},$$

extending the Pythagorean harmony proportion and confirming the adequacy of the generalized means, allows to conclude the suitable complementariness of A_f and H_f with respect to $\mathcal{G}$, and determine the convergence of sequence of the iterates of the mean-type mapping (A_f, H_f) to $(\mathcal{G}, \mathcal{G})$.

Moreover, for each of the classical symmetric means $\mathcal{A}$, $\mathcal{H}$, $\mathcal{G}$ and for the above generalized means A_f, G_f, H_f we prove the existence and uniqueness of the respective complementary mean, we give its explicit formula, as well as the limit of the sequence of iterates of the relevant mean-type mappings. We also give solutions of some special cases of the invariance equation involving the above generalized classical weighted means.

Karol Gryszka 🌍 *On the divergence of continuous functions* (joint work with Paweł Pasteczka)

For a family of continuous functions $f_1, f_2, \ldots : I \to \mathbb{R}$ (I is a fixed closed interval) with $f_1 \leq f_2 \leq \ldots$ define a set

$$I_f := \left\{x \in I : \lim_{n\to\infty} f_n(x) = +\infty\right\}.$$

We study the properties of the family of all admissible I_f-s and the family of all admissible I_f-s under the additional assumption

$$\lim_{n\to\infty} \int_x^y f_n(t)\,dt = +\infty \qquad \text{for all } x, y \in I \text{ with } x < y.$$

The origin of this problem is the limit behaviour of quasiarithmetic means.

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Eszter Gselmann 🌍 *Polynomial identities satisfied by generalized polynomials*

The study of additive mappings from a ring into another ring which preserve squares was initiated by G. Ancochea in [1] in connection with problems arising in projective geometry. Later, these results were again strengthened by (among others) Kaplansky [4] and Jacobson-Rickart [3]. It was G. Ancochea who firstly dealt with the connection of Jordan homomorphisms and homomorphisms, see [1] and its related results [3], [4], [5]. Roughly speaking, all the investigated problems were of the following form: Let R and R' be (commutative) rings and $P \in R[x]$

and $Q \in R'[x]$ be polynomials. Determine all those additive functions that also satisfy

$$f\big(P(x)\big) = Q\big(g(x)\big)$$

for all $x \in \mathbb{R}$. Clearly, this problem is meaningful if instead of additive functions, generalized monomials are considered. According to this remark, the talk will be devoted to the following problem: let n be a positive integer, $\mathbb{F}$ be a field and $P \in \mathbb{F}[x]$ and $Q \in \mathbb{C}[x]$ be polynomials. Determine those generalized polynomials $f, g\colon \mathbb{F} \to \mathbb{C}$ of degree n that also fulfill equation

$$f\big(P(x)\big) = Q\big(g(x)\big)$$

for each $x \in \mathbb{F}$. As it turns out, the difficulty of such problems heavily relies on that we consider the above equation for generalized polynomials or for (normal) polynomials. Therefore, firstly we study the connection between these two notions.

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Grzegorz Guzik *On geometric rate of convergence for generalized random iterations* (joint work with Rafał Kapica)

Fix $m \in \mathbb{N}$. Let (X, ϱ) be a a complete and separable metric space and $(\Omega, \Sigma, \mathbb{P})$ be a probability space. We consider a *generalized iterated function system with constant probabilities* (GIFSP), i.e. $\mathbb{P}$-continuous family $\mathcal{F} = \{f_\omega\colon X^m \to X :\ \omega \in \Omega\}$.

Given an infinite word $\omega^\infty = (\omega_n)_{n\in\mathbb{N}} \in \Omega^\infty$ we define a (random) ω^∞-*trajectory* of a GIFSP $\mathcal{F}$ starting from $x_1, \ldots, x_m \in X$ as a sequence $(x_n)_{n\in\mathbb{N}}$ of points from X defined by

$$x_{n+m} = f_{\omega_n}(x_n, \ldots, x_{n+m-1}) \qquad \text{for } n \in \mathbb{N}. \tag{1}$$

The above formula is a generalization of well known iteration process for so-called random valued functions studied by many authors on the ground of functional equations as well as operator theory.

We study the long-time behavior of random trajectories (1) of GIFSP with arbitrary number of transformations. To do this we discover the theory of limiting behavior for the pretty large class of operators of Markov-type acting on the product of spaces of Borel measures on X. We show the simple criterion on the existence of the unique invariant measure for such operator (asymptotic stability).

Moreover, we prove that this unique measure is a probability one with the first moment finite, it attracts all trajectories of distributions under the action of the operator in the sense of the Hutchinson-Wasserstein norm as well in the weak topology. The rate of convergence is the geometric one.

One can apply this result for studying asymptotic behavior of solutions of stochastic difference equations with multiple delays, for example in the case when such equation is a discretization of stochastic differential Itô one.

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László Horváth 🌍 *Refinements of the integral Jensen's inequality generated by finite or infinite permutations*

There are a lot of papers dealing with applications of the so called cyclic refinement of the discrete Jensen's inequality. A significant generalization of the cyclic refinement, based on combinatorial considerations, has recently been discovered. In the present talk we give the integral versions of these results. On the one hand, a new method to refine the integral Jensen's inequality is developed. Furthermore, the result contains some recent refinements of the integral Jensen's inequality as elementary cases. Finally, some applications to the Fejér inequality (especially the Hermite-Hadamard inequality), quasi-arithmetic means and f-divergences are presented.

Eliza Jabłońska *Remarks on classes $\mathcal{A},\mathcal{B},\mathcal{C}$*

Ger and Kuczma [4] introduced the following three families of sets in X:

- $\mathcal{A}(X)$ of all subsets $T \subset X$ such that any mid-convex function $f\colon D \to \mathbb{R}$ defined on a convex open subset $D \subset X$ containing T is continuous if $\sup f(T) < \infty$;
- $\mathcal{B}(X)$ of all subsets $T \subset X$ such that any additive function $f\colon X \to \mathbb{R}$ with $\sup f(T) < \infty$ is continuous;
- $\mathcal{C}(X)$ of all subsets $T \subset X$ such that any additive function $f\colon X \to \mathbb{R}$ is continuous if the set $f(T)$ in bounded in $\mathbb{R}$.

It is clear that

$$\mathcal{A}(X) \subset \mathcal{B}(X) \subset \mathcal{C}(X). \tag{1}$$

By the example of Erdős [2] (discussed in [4]) the classes $\mathcal{B}(X)$ and $\mathcal{C}(X)$ are not equal even if $X = \mathbb{R}^n$, $n \in \mathbb{N}$. On the other hand, Ger and Kominek [3] proved that $\mathcal{A}(X) = \mathcal{B}(X)$ for any Baire topological vector space X. In particular, $\mathcal{A}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}^n)$ for every $n \in \mathbb{N}$ (cf. [5]).

There are lots of papers devoted to the problem of recognizing sets in the families $\mathcal{A}(X)$, $\mathcal{B}(X)$ or $\mathcal{C}(X)$. Some recent results in [1] allow us to give some further examples of sets which are (not) in $\mathcal{A}(X)$, $\mathcal{B}(X)$ or $\mathcal{C}(X)$.

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Justyna Jarczyk *On generalized Archimedes-Borchardt algorithm* (joint work with Witold Jarczyk)

We investigate convergence and invariance properties of the generalized Archimedes-Borchardt algorithm. The main tool is reducing the problem to an appropriate Gauss iteration process.

Witold Jarczyk *Form of semiflows of pairs of weighted quasi-arithmetic means* (joint work with Dorota Głazowska, Justyna Jarczyk and Janusz Matkowski)

We start with a reformulation of Theorem 4 from the paper [1], more friend for our consecutive considerations than the original one. It provides a clear description of a general form of semiflows of pairs of weighted quasi-arithmetic means, with time running through the set $\mathbb{D}_+$ of all positive dyadic numbers (see Theorem 1). Because of the density of $\mathbb{D}_+$ in $(0, +\infty)$ we obtain the form of all continuous semiflows of such a type, over $(0, +\infty)$ (see Corollary 2).

In the second part of the talk we present an equivalent version of Theorem 1 from the paper [2] and observe that this statement runs contrary to Corollary 2. The reasons of such a situation are presented.

Finally we ask if the assumption of the continuity of semiflow is essential for the validity of Corollary 2.

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Sándor Jenei *Algebraic methods in functional equations – beyond the regularity conditions*

In [2, 3] it has been shown how algebraic and geometric methods can be used to solve an open problem posed in [1]. We shall shortly describe an example which demonstrates that when considering associative functions, algebraic methods can go much beyond even the most general regularity condition which is assumed in functional equations, namely the continuity of the solution, and one can still obtain all solutions. Moreover, the domain of the functions in our example is not necessarily a real interval either, it can be any linearly (totally) ordered set. The applied techniques (the theory of residuated semigroups) often work on lattices and even on partially ordered sets.

Under the following conditions we describe all solutions of the associativity equation

$$F\big(x, F(y,z)\big) = F\big(F(x,y), z\big)$$

- for $x, y \in X$ such that $F(x,y) = F(y,x)$,
- the domain of the function F is $X \times X$, X is any linearly ordered set,
- there exists $t \in X$ such that $F(x,t) = x$,
- for any $x, y \in X$ there exists the greatest element $F_{\rightarrow}(x,y)$ of the set $\{z \in X : F(x,z) \leq y\}$
- for $x \in X$, $F_{\rightarrow}(F_{\rightarrow}(x,t),t) = x$,
- $F_{\rightarrow}(t,t) = t$.

Or using an infix notation $x \circledast (y \circledast z) = (x \circledast y) \circledast z$, where

- $\circledast$ is a binary operation over a linearly ordered set X,
- for $x, y \in X$, $x \circledast y = y \circledast x$,
- there exists $t \in X$ such that $x \circledast t = x$,
- for any $x, y \in X$ there exists the greatest element $x \rightarrow_{\circledast} y$ of the set $\{z \in X : x \circledast z \leq y\}$,
- for $x \in X$, $(x \rightarrow_{\circledast} t) \rightarrow_{\circledast} t = x$,
- $t \rightarrow_{\circledast} t = t$

The fourth items say that F (resp. $\circledast$) is residuated. It implies that $\circledast$ is monotone increasing, and hence it can be also regarded as left-continuity.

This talk is based on the article [4].

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Soon-Mo Jung 🌍 *Approximate isometries on bounded sets*

More than 20 years after Fickett [2] attempted to prove the Hyers-Ulam stability of isometries defined on a bounded subset of $\mathbb{R}^n$ in 1982, Alestalo et al. [1] and Väisälä [4] improved the Fickett's theorem significantly. In this lecture, using a more intuitive and efficient approach, first used in the paper [3], we improve Fickett's theorem by proving the Hyers-Ulam stability of isometries defined on a bounded subset of $\mathbb{R}^n$.

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Gergely Kiss *Characterization of quasi-arithmetic means without regularity* (joint work with Pál Burai and Patrícia Szokol)

In this talk we recall the classical result of Aczél for the characterization of quasi-arithmetic means. We analysis the regularity assumption of this construction and we show that every bisymmetric, partially strictly monotonic, reflexive and symmetric function $F\colon I^2 \to I$ is continuous, where I is a proper subinterval of $\mathbb{R}$. This talk is based on the article [1] and the book [2].

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Grzegorz Kleszcz *On a functional-difference inclusion induced by some iterated function system* (joint work with Grzegorz Guzik)

In [1, Part II, Chapter 1] the following functional-difference equations with continuous-time argument

$$x(t+1) = f\big(x(t)\big) \tag{1}$$

is considered. In general, here above f is a given continuous function of some compact space X into itself and $x\colon [0,\infty) \to X$ is unknown. One can see that solutions of (1) depend on an arbitrary function. More precisely, having a function $\varphi\colon [0,1] \to X$ and putting

$$x|_{[0,1]} := \varphi \tag{2}$$

one can get uniquely the solution x on the whole semiline $[0,\infty)$ by the formula

$$x(t+n) = f^n \circ \varphi(t) \qquad \text{for } n \in \mathbb{N}. \tag{3}$$

In particular, if φ is a continuous function and the initial condition

$$\varphi(1) = f\big(\varphi(0)\big). \tag{4}$$

is assumed, the solution x of (1) is continuous.

The most interesting aspect is a study of asymptotic behaviors of graphs of a sequence $(f^n \circ \varphi)$ of compositions of iterates f^n with an 'observable' function φ in a proper Hausdorff metric or the topology of uniform convergence. In many cases such graphs tend to the graph of some set-valued function (with closed graph) rather than to the graph of a single-valued continuous function. Moreover, it is remarkable that usually the asymptotic behavior of such a sequence of graphs strongly depends on a choice of a function φ.

In various models of real systems the given function is not uniquely determined but either can be chosen from some family (finite or not) by a deterministic or random algorithm, or it is determined with some error. Hence it seems quite reasonable to consider the following inclusion

$$x(t+1) \in f\big(x(t)\big) \tag{5}$$

instead of the equality (1). Here $f\colon X \rightsquigarrow X$ is a given set-valued function which can represent a union of all functions possibly chosen.

In the present talk we consider the inclusion (5) with a set-valued function f generated by an iterated function system, i.e. a finite family of continuous mappings of X into itself, which is assumed to possess an attractor (a fractal set). Our main result says that under quite typical assumptions on a given iterated function system we get a simple asymptotic behavior of graphs of compositions $f^n \circ \varphi$, $n \in \mathbb{N}$ which is, surprisingly, independent on a choice of an 'observable' φ.

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Tomasz Kochanek *Approximately order zero maps between C*-algebras*

A bounded linear operator $\phi\colon A \to B$ acting between two C*-algebras is called an ε-order zero map, provided that $\|\phi(x)\phi(y)\| \le \varepsilon\|x\|\|y\|$ for all positive $x, y \in A$ with $xy = 0$. This terminology arises naturally from the theory of order zero maps and nuclear dimension of C*-algebras developed by Winter and Zacharias in [2] and [3]. In that theory, completely positive order zero maps (i.e. preserving zero products of positive elements) correspond to usual partitions of unity in the commutative setting, and they serve as building blocks for the notion of nuclear dimension which is a noncommutative analogue of the topological covering dimension.

During the talk, we will report on selected results from [1] which yield some structural properties of ε-order zero maps concerning approximate Jordan-like equations and almost commutation relations. The main goal is to decide, for a given pair (A, B) of C*-algebras, whether and under what conditions any ε-order zero map $\phi\colon A \to B$ can be approximated by an approximate Jordan *-homomorphism, with both errors of approximation depending only on $\|\phi\|$ and ε. As we shall explain, this is for instance possible in the case where the codomain algebra B is finite-dimensional. In particular, if $B = \mathbb{M}_n(\mathbb{C})$ is the $n \times n$ matrix algebra, then for an arbitrary C*-algebra A, any $0 < \varepsilon \leq 1$, and any positive ε-order zero map $\phi\colon A \to \mathbb{M}_n(\mathbb{C})$, there exist a corner C*-subalgebra $C \subseteq \mathbb{M}_n(\mathbb{C})$ and a bounded linear operator $\Phi\colon A_1 \to C$ defined on the unitization of A satisfying

$$\|\phi - \Phi\| \leq 37\|\phi\|^{4/5}\varepsilon^{1/16}$$

and such that either $\Phi = 0$ or $\Phi(1)$ is invertible in C in which case the operator $\Phi(1)^{-1}\Phi(\,\cdot\,)$ is an approximate Jordan *-homomorphism with an error δ estimated by

$$\delta = \mathcal{O}(256^n)\|\phi\|\varepsilon^{1/16}.$$

A similar result is valid in the case where A is commutative, B is arbitrary and $\phi\colon A \to B$ is assumed to be a surjective self-adjoint ε-order zero map. In this situation however, an approximating operator takes values in a corner subalgebra of the bicommutant B''.

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Zbigniew Leśniak 🌍 *On a construction of iterative roots of a Brouwer homeomorphism*

We present a method of construction of continuous orientation preserving iterative roots of a Brouwer homeomorphism under the assumption that for this homeomorphism there exists a family of pairwise disjoint invariant lines covering the plane which satisfies a matching property. The roots are constructed recursively for a family of maximal parallelizable regions, each of which is the union of the invariant lines. The matching property is used to obtain the continuity of the constructed iterative roots.

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Radosław Łukasik *Vector-valued invariant means and projections from the bidual space*

Invariant means on amenable groups are an important tool in many parts of mathematics, especially in harmonic analysis. Invariant means and their generalizations for vector-valued functions play also an important role in the stability of functional equations and selections of set-valued functions.

Some generalization of the invariant mean for vector-valued functions was investigated in [2] and the existence of such means is connected with reflexive spaces.

Some generalized definition of an invariant mean has been used by many mathematicians as a folklore (e.g. by A. Pełczynski [5]). The explicit form of this definition we can find, e.g. in the work of R. Ger [3]. The space of all bounded functions from a set S into a Banach space X is denoted by $\ell_\infty(S,X)$.

Definition
Let $(S,+)$ be a left [right] amenable semigroup, X be a real Banach space. A linear map $M\colon \ell_\infty(S,X)\to X$ is called left [right] X-valued invariant mean if

$$\begin{aligned}&||M||\le 1,\\&M(c\mathbb{1}_S)=c,\ c\in X,\\&M({}_af)=M(f),\ a\in S, f\in\ell_\infty(S,X),\\&[M(f_a)=M(f),\ a\in S, f\in\ell_\infty(S,X),]\end{aligned}$$

where

$$\begin{aligned}&{}_af(x)=f(a+x),\ a,x\in S,\ f\in\ell_\infty(S,X),\\&[f_a(x)=f(x+a),\ a,x\in S,\ f\in\ell_\infty(S,X).]\end{aligned}$$

If M is a left and right invariant mean, then M is called X-valued invariant mean.

If in the above definition norm of map M is equal at most $\lambda\ge 1$, then M is called X-valued invariant λ-mean.

The existence of such invariant means for a fixed Banach space and for all amenable semigroups has been studied by H.B. Domecq [1, Theorem 2] (his paper has a gap in the proof which was corrected by T. Kania [4]).

In this talk we will show a connection between the existence of X-valued invariant λ-means on X and projections from subspaces of bidual space X^{**} (with large enough density) onto X.

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Arpita Mal 🌍 *Extreme contraction from the perspective of multi-smoothness*

In this talk we explore a relation between extreme contraction and multi-smoothness of bounded linear operators on polyhedral Banach spaces. Using this relation we describe extreme contractions on two-dimensional polygonal Banach spaces. We provide a large class of Banach spaces $\mathbb{X}, \mathbb{Y}$ such that for each extreme contraction $T\colon \mathbb{X} \to \mathbb{Y}$ there is an extreme point $x \in \mathbb{X}$ such that Tx is an extreme point of $\mathbb{Y}$. We also show how the discussed approach helps us to explicitly compute the number of extreme contractions on some Banach spaces. This talk is based on the paper [1].

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Daniela Marian 🌍 *On Hyers-Ulam-Rassias stability of certain partial differential equations and integral equations*

In the first part we will present the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability for the Darboux-Ionescu problem. This part is based on [2].

In the second part we will present some results for a Volterra-Hammerstein integral equation with modified arguments: existence and uniqueness, integral inequalities, monotony and Ulam-Hyers-Rassias stability. This part is based on the article [1].

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Joanna Markowicz *Uniform convexity of direct sums and interpolation spaces*

A natural question in the theory of direct sums and interpolation spaces is whether given property of Banach spaces can be preserved under passing to a direct sum or an interpolation space.

In this talk, we will recall the well known theory of uniform convexity.

Subsequently, we will present the estimation of the modulus of convexity and characteristic of convexity of a general direct sum of Banach spaces.

Next, interpolation spaces obtained with the use of the general discrete interpolation method based on an abstract space with an unconditional basis will be introduced. We will give also conditions which guarantee such interpolation spaces be uniformly convex.

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Janusz Matkowski 🌍 *Remarks on fixed point theorem*

A generalization of Browder-Göhde-Kirk fixed point theorem and its application will be presented.

Rayene Menzer 🌍 *Alternative equations for quadratic functions* (joint work with Zoltán Boros)

Kominek, Reich and Schwaiger [2] investigated additive functions that satisfy the additional equation

$$f(x)f(y) = 0 \tag{1}$$

for every $(x, y) \in D$, considering various subsets D of $\mathbb{R}^2$, involving, for instance, the graphs of polynomials and the unit circle.

The result for the unit circle was extended to the case when f is a generalized polynomial by Boros and Fechner [1]. On the other hand, Kutas [3] proved the existence of a not identically zero additive function f that satisfies (1) whenever $xy = 1$.

Motivated by these preliminaries, we investigate quadratic functions f that satisfy (1) for every $(x, y) \in D$, considering various subsets D of $\mathbb{R}^2$. For the case when D is the graph of a polynomial, we establish the following statement.

Theorem 1
If $n \in \mathbb{N}$, $a_j \in \mathbb{R}$ $(j = 0, 1, \dots, n-1)$ and the quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the equation

$$f(x) \cdot f\left(x^n + \sum_{k=0}^{n-1} a_k x^k\right) = 0$$

for every $x \in \mathbb{R}$, then $f(x) = 0$ identically.

While Kutas' counterexample for the hyperbola $xy = 1$ can be easily extended to quadratic functions, we can establish a positive result for another hyperbola.

Theorem 2
If $f\colon \mathbb{R} \to \mathbb{R}$ *is a quadratic function fulfilling* $f(x)f(y) = 0$ *whenever* $x^2 - y^2 = 1$, *then* $f(x) = 0$ *identically for every* $x \in \mathbb{R}$.

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Flavia-Corina Mitroi-Symeonidis *Redistributing algorithms and Shannon's entropy* (joint work with Eleutherius Symeonidis)

The mathematical problems discussed in our talk arise in the framework of time series analysis, namely in the analysis of temperature values measured during compartment fire experiments. The permutation entropy is used for such analysis, and the underlying permutations were established using redistributing algorithms. We investigate how Shannon's entropy changes when the distribution probability is modified by means of redistributing algorithms.

Let a probability distribution $P = (p_1, \dots, p_n)$ be such that it has at least k nonzero components $p_{n-k+1}, \dots, p_n$ $(1 \le k \le n-1)$. We are interested to find the best (largest) domain $\Delta \subset \mathbb{R}^k$ such that

$$H\left(p_1, \dots, p_{n-k}, p_{n-k+1}, \dots, p_n\right) \ge H\left(0, \dots, 0, 1/k, \dots, 1/k\right)$$

for all P with $(p_{n-k+1}, \dots, p_n) \in \Delta$.

We mention the following immediate consequences of interest, which we assume that are already known (we have not found any reference for them yet):

Let n, k and p be fixed. The entropy $H\left(p_1, \dots, p_{n-k}, p, \dots, p\right)$ attains its maximum at

$$p_1 = \ldots = p_{n-k-1} = p_{n-k} = \frac{1-kp}{n-k},$$

and its minimum if there is $j = 1, \dots, n-k$ such that $p_j = 1 - kp$.

Mădălina Moga *On some properties of Meir-Keeler operators*

The first purpose of this talk is to prove that, in a Banach space X, a Meir-Keeler operator f (singlevalued or multivalued) is a norm-contraction. Then, using this result, we will give sufficient conditions assuring that the field $1_X - f$, generated by f, is surjective. It also contains surjectivity theorems for singlevalued and, respectively, multivalued Meir-Keeler operators. In the last part we present an application for the previous results. Our results generalize some well-known theorems of this type for Banach/Nadler type contractions, see [5] and [2], as well as other results of this type for generalized contractions, see also [1], [3], [6], [7].

We will present a new surjectivity theorem for a singlevalued Meir-Keeler operator. The approach is based on the norm-contraction operator theory and on the well-known theorem of A. Granas [4].

Theorem 1
Let $(X, \|\cdot\|)$ be a Banach space and Y be a nonempty, closed and convex subset of X. Let us consider the operator $f\colon Y \to Y$ satisfying the following assumptions:

1. *f restricted to any bounded set in Y is compact;*
2. *f is a Meir-Keeler operator.*

Then $1_Y - f$ is a surjective operator.

Our main result concerning the surjectivity of the multivalued field generated by a Meir-Keeler operator is the following theorem.

Theorem 2
Let $(X, \|\cdot\|)$ be a Banach space and Y be a nonempty, closed and convex subset of X. Let us consider the multivalued operator $T\colon Y \to P_{cp,cv}(Y)$ satisfying the following assumptions:

1. *for each $A \in P_b(X)$ the set $T(A)$ is relatively compact;*
2. *T is a Meir-Keeler operator.*

Then, the field $1_Y - T$ generated by T is surjective.

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Lajos Molnár *On the order determining property of the norm of a Kubo-Ando mean in operator algebras*

Recently, several papers have been published concerning preservers of norms of means on function algebras and operator algebras. In our approach to the study of those transformations and in the proofs of our results the key point was an order determining property of the norms of the considered means.

Motivated by that, in a current paper we have initiated the investigation of the mentioned property in a general setting. The aim of this talk is to report on the obtained results concerning the question of when the operator norm of a Kubo-Ando mean σ determines the order on the positive definite cone of an operator algebra. More precisely, the question asks when, for any given pair A, B of positive definite elements of the underlying algebra, we have that $A \leq B$ holds if and only if $\|A\sigma X\| \leq \|B\sigma X\|$ is valid for all positive definite elements X. Our study is not completed, besides presenting results we formulate open problems.

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Veerapazham Murugan 🌍 *Iterative root problem for continuous functions of non-monotonicity height 1* (joint work with Rajendran Palanivel)

In this paper, we define the non-monotonicity height for any continuous self-maps on a compact interval and prove some interesting properties of the non-monotonicity height of non-PM functions. We obtain necessary and sufficient conditions for the existence of iterative roots for a class of continuous functions of non-monotonicity height 1.

Hemant Kumar Nashine 🌍 *Fixed point results on relational metric spaces with applications to Hyers-Ulam stability and nonlinear matrix equations*

Throughout this talk, we will develop some fixed point results for $\mathcal{FG}$-contractive mappings on complete metric spaces equipped with any binary relation (not necessarily a partial order). Additionally, the Ulam-Hyers stability, well-posedness, and limit shadowing properties of this issue are examined. We use the aforementioned fixed point result to find the solution to a non-linear matrix equation of the form $\mathcal{X} = \mathcal{Q} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{X}) \mathcal{A}_i$, where $\mathcal{Q}$ is a Hermitian positive definite matrix, $\mathcal{A}_i^*$ stands for the conjugate transpose of an $n \times n$ matrix $\mathcal{A}_i$ and $\mathcal{G}$ is an order-preserving continuous mapping from the set of all Hermitian matrices to the set of all positive definite matrices such that $\mathcal{G}(O) = O$. We explore the necessary and sufficient criteria for the existence of a unique positive definite solution to a particular matrix problem. Through several demonstrations using graphical representations, we demonstrate the fixed-point conclusions and the relevance of related work, as well as the convergence analysis of non-linear matrix equations.

Definition 1 ([1])
The collection of all functions $\mathcal{F}\colon \mathbb{R}_+ \to \mathbb{R}$ satisfying:

($\mathcal{F}_1$) $\mathcal{F}$ is continuous and strictly increasing;

($\mathcal{F}_2$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\lim\limits_{n\to\infty} \xi_n = 0$ iff $\lim\limits_{n\to\infty} \mathcal{F}(\xi_n) = -\infty$,

will be denoted by $\mathbb{F}$.

The collection of all pairs of mappings $(\mathcal{G},\beta)$, where $\mathcal{G}\colon \mathbb{R}_+ \to \mathbb{R}$, $\beta\colon \mathbb{R}_+ \to [0,1)$, satisfying:

($\mathcal{F}_3$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\limsup_{n\to\infty} \mathcal{G}(\xi_n) \geq 0$ iff $\limsup_{n\to\infty} \xi_n \geq 1$;

($\mathcal{F}_4$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\limsup_{n\to\infty} \beta(\xi_n) = 1$ implies $\lim_{n\to\infty} \xi_n = 0$;

($\mathcal{F}_5$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\sum_{n=1}^{\infty} \mathcal{G}(\beta(\xi_n)) = -\infty$,

will be denoted by $\mathbb{G}_\beta$.

We call $(\mathcal{X},\mathfrak{R})$ a relational set if (i) $\mathcal{X} \neq \emptyset$ is a set and (ii) $\mathfrak{R}$ is a binary relation on $\mathcal{X}$. In addition, if $(\mathcal{X},d)$ is a metric space, we call $(\mathcal{X},d,\mathfrak{R})$ a relational metric space (*RMS*, for short).

Definition 2
Let $(\mathcal{X},d,\mathfrak{R})$. A mapping *RMS* and $\mathcal{T}\colon \mathcal{X} \to \mathcal{X}$ is said to be a $\mathcal{F}\mathcal{G}$-contractive mapping, if there are $\mathcal{F} \in \mathbb{F}$ and $(\mathcal{G},\beta) \in \mathbb{G}_\beta$, such that for $(\nu,\vartheta) \in \mathcal{X}$ with $(\nu,\vartheta) \in \mathfrak{R}^*$,

$$\mathcal{F}(d(\mathcal{T}\nu,\mathcal{T}\vartheta)) \leq \mathcal{F}(\Theta(\nu,\vartheta)) + \mathcal{G}(\beta(\Theta(\nu,\vartheta))),$$

where

$$\Theta(\nu,\vartheta) = \max\left\{d(\nu,\vartheta), d(\nu,\mathcal{T}\nu), d(\vartheta,\mathcal{T}\vartheta), \frac{d(\nu,\mathcal{T}\vartheta)+d(\vartheta,\mathcal{T}\nu)}{2}\right\}$$

and $\mathfrak{R}^* = \{(\nu,\vartheta) \in \mathfrak{R} :\ \mathcal{T}\nu \neq \mathcal{T}\vartheta\}$. We denote by $(FG)_\mathfrak{R}$ the collection of all $\mathcal{F}\mathcal{G}$-contractive mappings on $(\mathcal{X},d,\mathfrak{R})$.

Theorem 1
Let $(\mathcal{X},d,\mathfrak{R})$ be an RMS and $\mathcal{T}\colon \mathcal{X} \to \mathcal{X}$. Suppose that the following conditions hold:

(i) $\mathfrak{X}(\mathcal{T},\mathfrak{R}) \neq \emptyset$;

(ii) $\mathfrak{R}$ *is $\mathcal{T}$-closed and $\mathcal{T}$-transitive;*

(iii) $\mathcal{X}$ *is $\mathcal{T}$-$\mathfrak{R}$-complete;*

(iv) $\mathcal{T} \in (FG)_\mathfrak{R}$;

(v) $\Im$ *is $\mathfrak{R}$-continuous or*

(vi) $\mathfrak{R}$ *is d-self-closed.*

Then there exists a fixed point $\nu_* \in \mathcal{X}$ of $\mathcal{T}$.

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Kazimierz Nikodem *Remarks on set-valued means*

Let X be a real vector space and D be a convex nonempty subset of X. Denote by $S(D)$ the family of all nonempty subsets of D. We say that a function $M\colon D^n \to S(D)$ is a *set-valued mean* if

$$M(x_1, \dots, x_n) \subset \operatorname{conv}\{x_1, \dots, x_n\},$$

for all $x_1, \dots, x_n \in D$.

Set-valued counterparts of the arithmetic, quasi-arithmetic and Lagrangian means are investigated and various properties of them are presented. In particular, the following result is presented.

Theorem
Let $f, g\colon I \to J$ be strictly increasing functions such that f is concave and g is convex. Assume that $f \le g$ on I and $F(x) = [f(x), g(x)]$ holds for all $x \in I$. Then the map $A_F\colon I^n \to S(I)$ given by

$$A_F(x_1, \dots, x_n) = F^{+}\left(\frac{1}{n}\sum_{i=1}^{n} F(x_i)\right), \qquad x_1, \dots, x_n \in I,$$

and the map $L_F\colon I^2 \to S(I)$ given by

$$L_F(x_1, x_2) = \begin{cases} F^{+}\left(\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} F(x)dx\right), & \text{if } x_1 \neq x_2 \\ \{x_1\}, & \text{if } x_1 = x_2 \end{cases}$$

are set-valued means.

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Chisom Prince Okeke *On some new functional equations characterizing polynomial functions*

This is a continuation of the talk presented by Maciej Sablik and containing the theoretical results of the functional equations belonging to the class (1),

$$F(x+y) - F(x) - F(y) = \sum_{i=1}^{m} (a_i x + b_i y) f(\alpha_i x + \beta_i y) \tag{1}$$

where $F, f\colon \mathbb{R} \to \mathbb{R}$ are unknown functions and for each $i \in \{1, \dots, m\}$ the numbers $a_i, b_i \in \mathbb{R}$ and $\alpha_i, \beta_i \in \mathbb{Q}$ are given.

We observed that solving even simple equations belonging to class (1) needs a long and tiresome calculation. Therefore, we developed an algorithm written in the computer algebra system Maple which takes into account the theoretical results to determine the polynomial solutions of the functional equations belonging to the class (1) (cf. also [1], [2] and [3]).

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Andrzej Olbryś *On characterization of affine differences and related forms*

In the talk we consider the problem of characterizing affine differences and some related expressions. We use functional equations to characterize expressions $\omega\colon X \times X \times [0,1] \to Y$ of the form

$$\omega(x,y,t) = tf(x) + (1-t)f(y) - f(tx + (1-t)y), \qquad x, y \in X,\ t \in [0,1],$$

where $f\colon X \to Y$ is an arbitrary function and X, Y are real linear spaces.

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Adam Ostaszewski *Tilted solutions of the Gołąb-Schinzel functional equation in Banach algebras* (joint work with Nicholas Hugh Bingham)

In $\mathbb{R}$ the Gołąb-Schinzel functional equation, briefly: (GS), has continuous solutions affine:

$$S(x) = 1 + \rho x \text{ on } G_S := \{x :\ 1 + \rho x > 0\} \subseteq \mathbb{R}.$$

Question: For S with values in a Banach algebra $\mathbb{A}$: what form does a solution S have on

$$\mathbb{G}_S^*(\mathbb{A}) := \{x \in \mathbb{A} :\ S(x) \in \mathbb{A}^{-1}\}, \text{ or on } \mathbb{G}_S(\mathbb{A}) := \{x \in \mathbb{A} :\ S(x) \in \mathbb{A}_1\},$$

for $\mathbb{A}^{-1}$ the invertible elements and $\mathbb{A}_1$ the connected component of the identity?

S is affine iff $\mathbb{A}$-differentiable (for $\mathbb{A}^{-1}$ *dense*): compare [2]. Below S is *Fréchet differentiable* with derivative $S'(0)u$ at 0 in direction u.

Define the following

$N(x) := S(x) - 1_{\mathbb{A}} - \rho x$	for $\rho := S(1_{\mathbb{A}}) - 1_{\mathbb{A}}$ $(x \in \mathbb{G}_S^*,$ assuming $S(1_{\mathbb{A}}) \in \mathbb{A}^{-1})$;
$\lambda_u(t) := (e^{t\gamma(u)} - 1)/(e^{\gamma(u)} - 1)$	for $\gamma(u) := S'(0)u$ with L'Hospital convention;
$T(u) := u(e^{\gamma(u)} - 1)/\gamma(u)$	for $u \in \mathbb{A}$ (the 'exponential tilting' map).

Characterization Theorem (Curvilinear exponential homogeneity)
For S a Fréchet differentiable solution of (GS), N satisfies a (Levi-Civita-style) Goldie equation:

$$N(x + S(x)y) = N(x) + S(x)N(y), \qquad x, y \in \mathbb{G}_S^*.$$

Furthermore, for all $u \in \mathbb{A}$ with spectrum $\mathrm{spec}(\gamma(u))$ *not separating* 0 *from* ∞:

$$N(u(e^{t\gamma(u)} - 1)/\gamma(u)) = \lambda_u(t)N(u(e^{\gamma(u)} - 1)/\gamma(u)), \qquad t \geq 0,$$
$$N(tu) = t1_{\mathbb{A}}N(u) = tN(u), \qquad t \geq 0, \text{ when } 1 \in \exp(\mathrm{spec}(\gamma(u))).$$

So T exhibits *'curvilinear exponential homogeneity'* under N:

$$N(T(tu)) = \lambda_u(t)N(T(u)).$$

Corollary
$\mathcal{N} := \{x : S(x) = 1_{\mathbb{A}}\}$ *is linear if, for all large v, one of $\pm v = T(u)$ is soluble.*

Theorem (Maximal linear subspace characterization). *With assumptions as before,*

$$\mathcal{H} := \{x \in \mathbb{A} : (\forall t \in \mathbb{R})\ N(tx) = tN(x)\} = \{u \in \mathbb{A} : S'(0)u = 0\} \subseteq \mathcal{N}.$$

So if $\mathcal{N}$ is linear, then $\mathcal{N} = \mathcal{H}$ and N is linear on $\mathcal{N}$. For $\mathcal{M}$ complementary to $\mathcal{N}$ with $\pi_{\mathcal{M}}, \pi_{\mathcal{N}}$ corresponding projections:

$$N(x) = N(\pi_{\mathcal{N}}(x)) + N(\pi_{\mathcal{M}}(x)). \qquad (+)$$

Key Tools: $\mathbb{G}_S(\mathbb{A})$ a Popa group under $x \circ_S y := x + S(x)y$ and the symbolic calculus. See [1].

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Zdzisław Otachel *New advances on some inequalities in inner product spaces*

Probably, the most known inequality related to inner product spaces is Cauchy-Schwarz inequality which states that the module of inner product of two vectors is not greater than product of norms of that vectors. The inequality is closely related to other, more sophisticated, inequalities of that type as Bessel's, Grüss', Gram's or Hadamard's inequalities.

The theory of such inequalities plays an important role in modern mathematics together with numerous applications for Nonlinear Analysis, Approximation and Optimization Theory, especially Engineering Optimization, Numerical Analysis, Probability Theory, Statistics, Theoretical Physics and other fields. The Schwarz and Grüss inequalities have been frequently used for obtaining bounds or estimating the errors in various approximation formulas occurring in the above domains. Thus, any new advancement will have a number of important consequences in the mathematical fields where inequalities are basic tools. New versions of classic inequalities also stimulates further developments of the more general theory of functional equations and inequalities.

The presentation basing on papers [1]-[5] is mainly devoted to reverses of Cauchy-Schwarz's inequality and consequences for other related inequalities. Cauchy-Schwarz inequality is valid for arbitrary vectors of the space, whereas its reverses hold true under certain restrictions and usually define upper bounds for the product of (squared) norms of two vectors reduced by (squared) module of inner product of these vectors. Applications to reverse Bessel's inequality, the reverse triangle inequality for norms and Grüss type inequalities are given and refinements of the famous Hadamard's inequality about determinants are presented.

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Zsolt Páles *Limit theorems for deviation means of independent and identically distributed random variables* (joint work with Mátyas Barczy)

Motivated by the Kolmogorov Strong Law of Large Numbers and the Lindberg-Lévy Central Limit Theorem, furthermore by some recent results of de Carvalho [1] concerning quasiarithmetic means and Barczy-Burai [2] concerning Bajraktarević means, replacing the arithmetic mean of a sequence of indpendent and identically distributed random variables by a deviation mean (cf. [3]), we obtain generalizations of the above mentioned limit theorems.

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Paweł Pasteczka *On the Riemannian approach to integral means*

Adapting the notion of Riemann integral, for every weighted mean we define the corresponding lower and upper integral means. We show that such approach preserves most of properties of means, for example monotonicity and convexity/concavity.

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Kallol Paul *k-smoothness of operators on infinite-dimensional spaces*

In this talk we characterize k-smoothness of bounded linear operators defined between infinite-dimensional Hilbert spaces. The characterization of k-smoothness of bounded linear operators on arbitrary Banach space is still elusive. We try to address the problem in the setting of both finite and infinite-dimensional Banach spaces. This talk is based on the articles [1], [2].

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Adrian Petruşel *Existence and Stability Results in Fixed Point Theory* (joint work with Gabriela Petruşel)

The purpose of this talk is to present some new results in metric fixed point theory for single-valued and multi-valued operators. The case of graphical (orbital) contractions will be considered. We will discuss existence of the fixed point, as well as data dependence, well-posedness, Ostrowski stability property and Ulam-Hyers stability of the fixed point problem. In the case of the stability properties associated to the fixed point problem, the approach will put in the light the role of the retraction-displacement condition.

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Dorian Popa *Best constant in Ulam Stability*

In this talk I present results on the best Ulam constant for some functional equations and some linear operators.

Teresa Rajba *On the Raşa inequality for higher-order convex functions*

This talk is based on the article [2]. We study the following $(q-1)$-th convex ordering relation for q-th convolution power of the difference of probability distributions μ and ν,

$$(\nu-\mu)^{*q} \geq_{(q-1)cx} 0, \qquad q \geq 2,$$

and we obtain the theorem providing a useful sufficient condition for its verification. We apply this theorem for various families of probability distributions and we obtain several inequalities related to the classical interpolation operators. In particular, taking binomial distributions, we obtain a new, very short proof of the inequality given recently by Abel and Leviatan [1].

Theorem 1 ([1])
Let $q, n \in \mathbb{N}$, $q \geq 2$ and $x, y \in [0,1]$. If $f \in \mathcal{C}([0,1])$ is a $(q-1)$-convex function, then

$$\operatorname{sgn}(x-y)^q \sum_{\nu_1,\ldots,\nu_q=0}^{n} \sum_{j=0}^{q} (-1)^{q-j} \binom{q}{j} \Big(\prod_{i=1}^{j} p_{n,\nu_i}(x) \Big) \Big(\prod_{i=j+1}^{q} p_{n,\nu_i}(y) \Big) \times f\Big(\frac{\nu_1+\ldots+\nu_q}{qn}\Big) \geq 0, \tag{1}$$

where

$$p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \qquad 0 \leq i \leq n.$$

Inequality (1) is a generalization of inequality stated by Ioan Raşa as an open problem in [4] and proved in [3].

Theorem 2 ([3])
Let $n \in \mathbb{N}$ *and* $x, y \in [0,1]$. *Then*

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\left(p_{n,i}(x)p_{n,j}(x)+p_{n,i}(y)p_{n,j}(y)-2p_{n,i}(x)p_{n,j}(y)\right)f\left(\frac{i+j}{2n}\right)\geq 0$$

for all convex functions $f \in \mathcal{C}([0,1])$.

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Ioan Raşa 🌍 *A functional equation related to Appell polynomials and Heun functions* (joint work with Ana Maria Acu)

We are concerned with functional equations of the form

$$\omega(x)y_n'(x) = n\omega'(x)(y_n(x) - y_{n-1}(x)), \tag{1}$$

where ω is a given function on an interval I. First, using the theory of Appell polynomials we determine all the solutions of (1). Then we present equations of type (1) for which the solutions are log-convex. Some polynomial Heun functions are log-convex solutions of such equations. We investigate the properties of the involved Appell polynomials and apply the results to the associated Jakimovski-Leviatan operators. Our results extend and generalize results from [1]-[4].

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Lenka Rucká *Some results on distributionally chaotic points* (joint work with Francisco Balibrea)

The distributionally chaotic point (DC point for short) is introduced in [1] as a point whose arbitrarily small neighbourhood contains an uncountable distributionally chaotic set, which is bounded by a special envelope. In this talk we extend the result from [1] and we show that for continuous interval maps, positive topological entropy implies existence of uncountably many DC points. Also we show that this result cannot be extended to higher class of maps, particularly to continuous triangular maps of the square.

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Maciej Sablik *On some new functional equations characterizing polynomial functions* (joint work with Chisom Prince Okeke)

We are considering the following class of equations

$$F(x+y) - F(x) - F(y) = \sum_{i=1}^{m} (a_i x + b_i y) f(\alpha_i x + \beta_i y), \tag{1}$$

where $F, f\colon \mathbb{R} \to \mathbb{R}$ are unknown functions and for each $i \in \{1, \ldots, m\}$ the numbers $a_i, b_i \in \mathbb{R}$ and $\alpha_i, \beta_i \in \mathbb{Q}$ are given.

Equation (1) is a far going extension of the equation considered by Włodzimierz Fechner and Eszter Gselmann, namely

$$F(x+y) - F(x) - F(y) = xf(y) + yf(x), \tag{2}$$

which was studied by the authors in [1]. Together with Timothy Nadhomi and Tomasz Szostok we studied in [2] the following extension of (2),

$$\sum_{i=1}^{n} \gamma_i F(\alpha_i x + \beta_i y) = xf(y) + yf(x).$$

The results have been published in [2].

We present a method of solving equations of type (1), based on a Lemma contained in [2].

This is the theoretical part of the talk which has been divided into two parts. The second will be presented by Chisom Prince Okeke.

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Debmalya Sain 🌍 *Best approximations from the perspective of orthogonality*

We plan to discuss an application of Birkhoff-James orthogonality in studying the best approximation problem in Banach spaces. This allows us to obtain some interesting distance formulae for both compact operators and vectors. The advantage of our method over the classical duality principle, especially from a computational point of view, will also be discussed. This talk is based on my recent article [1]

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Adolf Schleiermacher *On even orthogonally additive mappings* (joint work with Justyna Sikorska)

In 1990 György Szabó proved: *If an orthogonality space* $(X, \perp)$ *in the sense of Rätz* [3] *and of dimension at least 3 over the field of real numbers admits a nontrivial even orthogonally additive mapping* $E\colon X \to Y$ *into some Abelian group* Y*, then* X *is equivalent to an inner product space (cf.* [4]*).* Using methods from projective geometry (cf. [1], [2]) we prove a stronger theorem consisting of several equivalent assertions about $(X, \perp)$ and valid for arbitrary ordered groundfields K. Note that Dilian Yang has proved in [5] an analogue of Szabó's theorem for the case of dimension two. However, this case requires different methods.

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Juan Matías Sepulcre 🌍 *A class of functional equations associated with almost periodic functions*

During this talk we shortly describe how to get a class of functional equations involving a countable set of terms, summed by the well known Bochner-Fejér summation procedure, which are closely associated with the set of Bohr's almost periodic functions. First, we discuss the case of a finite number of terms in connection with exponential polynomials, including the initial motivation with the partial sums of the Riemann zeta function. Secondly, in conjunction with the study of some particular properties and solutions, we progressively extend the expression of our functional equations to the case of a countable set of terms, and we finally show that the zeros of a prefixed almost periodic function determine analytic solutions of such a functional equation associated with it. This talk is based on the articles [1], [2], [3] and [4].

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Yong-Guo Shi 🌍 *Periodic points of asymmetric Bernoulli shifts*

In 1964, Sharkovskii [4] firstly introduced a special ordering on the set of positive integers. This ordering implies that if $p \triangleleft q$ and a self-map of a closed bounded interval has a point of period p, then it has a point of period q. The least number with respect to this ordering is 3. Thus, if a map has a point of period 3, then it has points of any periods. In 1975, the latter result was rediscovered by Li and Yorke [1]. Then numerous papers are devoted to the study of interval maps (see, e.g. [2], [3], [5] and references therein)

Consider asymmetric Bernoulli shift $F\colon [0,1] \to [0,1]$ with a parameter $0 < a < 1$, defined by

$$F(x) := \begin{cases} \frac{x}{a}, & 0 \le x \le a, \\ \frac{x-a}{1-a}, & a < x \le 1. \end{cases}$$

Given a positive integer n, one interesting question is how to find all n-periodic points of F. The other is how many n-periodic points of F has.

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Justyna Sikorska *On a general n-linear functional equation* (joint work with Anna Bahyrycz)

General linear functional equations have been studied for years. During the talk we shall discuss their counterpart for multivariable functions.

Let X, Y be linear spaces over a field $\mathbb{K}$ and $f\colon X^n \to Y$. For some fixed $a_{ji} \in \mathbb{K} \setminus \{0\}$, $C_{i_1 \dots i_n} \in \mathbb{K}$ for all $j \in \{1, \dots, n\}$, $i, i_k \in \{1, 2\}$, $k \in \{1, \dots, n\}$, we consider the following equation

$$f(a_{11}x_{11}+a_{12}x_{12}, \dots, a_{n1}x_{n1}+a_{n2}x_{n2}) = \sum_{1 \le i_1, \dots, i_n \le 2} C_{i_1 \dots i_n} f(x_{1i_1}, \dots, x_{ni_n}), \quad (*)$$

for all $x_{ji_j} \in X$, $j \in \{1, \dots, n\}$, $i_j \in \{1, 2\}$.

We determine the general solution of $(*)$ and as special cases we obtain some results for multi-Cauchy, multi-Jensen and multi-Cauchy-Jensen equations.

László Székelyhidi *Functional equations, moment functions and Bell polynomials* (joint work with Żywilla Fechner and Eszter Novák-Gselmann)

During this talk we present how to describe moment function sequences of higher rank on commutative groups and on polynomial hypergroups. This talk is based on the articles [2] and [3].

The description depends on certain functional equation systems which generalize some binomial-type functional equation systems considered and solved in [1]. For the general description we use Bell polynomials.

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Patricia Szokol *Hermite - Hadamard type inequality for certain Schur convex functions* (joint work with Pál Burai and Judit Makó)

In the presentation we investigate symmetric, continuous n-variable functions that satisfy a property, which can be considered as a Hermite-Hadamard type inequality. In the main result we prove that such functions are necessarily Jensen-convex. It turns out, that the key tool of the proof is a Korovkin-type approximation theorem, which will be presented in the talk, as well. Finally, we present some applications of our main theorem.

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Tomasz Szostok *On some class of means*

We consider means connected with quadrature rules of numerical integration. Then the functions which generate weighted arithmetic mean in this way are characterized and some inequalities between different means of this type are proved.

Lan Nhi To *Computer assisted investigation of the alienness of linear functional equations* (joint work with Attila Gilányi)

The concept of the alienness and the strong alienness of functional equations was introduced by J. Dhombres in [3]. The properties were investigated by many authors during the last years (cf., e.g. [4], [5], [7]).

In this talk, we present a computer program, developed in the computer algebra system Maple, which investigates the alienness and the strong alienness of linear functional equations of the type

$$\sum_{i=0}^{n+1} f_i(p_i x + q_i y) = 0, \qquad x, y \in X,$$

where n is a positive integer, $p_0, \dots, p_{n+1}$ and $q_0, \dots, q_{n+1}$ are rational numbers, X, Y are linear spaces and $f_0, \dots, f_{n+1} \colon X \to Y$ are unknown functions.

Our application is based on a computer program developed for the solution of linear functional equations of the type above, presented in [6] (cf. also [1] and [2]).

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Radu Truşcă *Some local fixed point results and applications for generalized contractions*

The purpose of this work is to present some local fixed point results, under weaker assumptions, for three types of generalized contractions: Ćirić-Reich-Rus, Chatterjea and Berinde. As applications we will present open mapping principles and continuation results for the three aforementioned classes of contractions. For example, let us consider the class of Ćirić-Reich-Rus generalized contractions.

Definition 1
Let (X, d) be a metric space. We say that an operator $f\colon X \to X$ is a Ćirić-Reich-Rus contraction if there exist $\alpha, \beta \in \mathbb{R}^+$ with $\alpha + 2\beta < 1$ such that we have

$$d(f(x), f(y)) \leq \alpha d(x, y) + \beta[d(x, f(x)) + d(y, f(y))] \qquad \text{for all } x, y \in X.$$

The local fixed point theorem related to Ćirić-Reich-Rus contractions is the following one.

Theorem 1
Let (X, d) be a complete metric space, $x_0 \in X$, a positive number r and let $f\colon B(x_0; r) \to X$ be a Ćirić-Reich-Rus type contraction. If

$$d(x_0, f(x_0)) < \frac{1-\alpha-2\beta}{1-\beta} r,$$

then the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to a point x^* which is a fixed point for the Ćirić-Reich-Rus contraction f. Moreover, the fixed point is unique.

As applications of our result, we will give open mapping principles and continuation results.

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Filip Turoboś *On preserving the relaxed polygonal inequality*

During the presentation we investigate the families of functions of the form $f\colon [0,+\infty) \to [0,+\infty)$ which preserve the relaxed polygonal inequality, introduced by Fagin et al. in [4]. Formally speaking, the definition of this concept is as follows:

Definition 1 (Relaxed polygonal inequality)
Let (X,d) be a semimetric space. If there exists $\gamma \geqslant 1$ such that

$$\forall_{n\in\mathbb{N}}\ \ \forall_{x_1,\ldots,x_n\in X}\quad d(x_1,x_n) \leqslant \gamma\cdot\sum_{k=1}^{n-1} d(x_k,x_{k+1}),$$

then we say that d satisfies the *relaxed polygonal inequality* (*rpi* in short) with the relaxation constant γ.

We investigate the families of functions which can be composed with any semimetric satisfying rpi to obtain another semimetric with this property (such mappings will be called (P)-preserving). In general, the concept of property-preserving functions was introduced as early as in 1947 by Sreenivasan [5], but most profound results were formulated by Slovakian mathematicians – Borsík and Doboš (e.g. [1], [2], [3]).

We would like to discuss two theorems which give equivalent conditions for f to be (P)*-preserving.*

Firstly, we introduce the easily-verifiable characterization for increasing (P)-preserving functions. Then we introduce some special finite tuples of positive real numbers which are meant to resemble distances between points in some semimetric space satisfying rpi. These so-called γ-polygons will be then used in our second characterization of (P)-preserving mappings. In particular, we will prove that every such mapping has to satisfy $f^{-1}\big[\{0\}\big] = \{0\}$ and for any $K_1 \geqslant 1$, there exists K_2 such that f maps every K_1-relaxed polygon to K_2-relaxed polygon.

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Szymon Wąsowicz 🌍 *New adaptive method of the approximate integration*

The classical Simpson's composite rule induces an adaptive method of the approximate integration, which stops an algorithm whenever some control expression containing the fourth derivative of an integrand is less from a desired precision of computation. Sometimes it is difficult to estimate this derivative. Then finding an algorithm which does not require any derivatives seems to be useful. Clenshaw and Curtis described in 1960 (without any proof) and Rowland and Varol examined in 1972 a method based precisely on the Simpson's rule. This is a starting point of our considerations. In the talk we introduce another method, which involves some convex combination of the Simpson's rule and the Chebyshev quadrature. The objects of our research are so-called convex functions of order three. We present the results of the numerical experiments which convince us that our adaptive method requires considerably less subpartitions of the interval of integration than the classical methods.

Bettina Wilkens *Algebraic methods in spectral synthesis on discrete abelian groups*

During this talk we present some applications of classical results of commutative algebra to established results and open problems in the area of spectral synthesis on discrete abelian groups. We can say more. Let G be an Abelian group and let V be a variety in $\mathcal{C}G$. *Spectral synthesis holds on* V if every subvariety is the closure of its finite-dimensional subvarieties. We can show something more:

Theorem
Let V be a variety on the Abelian group G. Let $R = \mathbb{C}G/V^\perp$. For a maximal ideal M of R, let $W_M = \bigcup_{n\in\mathbb{N}}(M^n)^\perp$. If spectral synthesis holds on V, then:

1. *The variety V is spanned by the varieties W_M, M running over the maximal ideals of R.*
2. *For each M, W_M has a subvariety U_M spanned by finite-dimensional subvarieties annihilated by a power of M.*
3. *For each M, W_M is the closure of U together with subvarieties of dimension bounded by some integer N depending only on M.*

Let U be any R-module. We say that U is *finitary* if, for every $x \in R$ and $u \in U$, there is $0 \neq f \in \mathbb{C}[x]$ such that $uf = 0$.

Lemma
If R has a faithful finitary module, then local spectral synthesis *holds on V.*

Local spectral synthesis (see [2]) is the property of being the closure of the linear span of functions restricting to polynomials on any finitely generated subgroup. We finish by discussing potential converses of the above lemma.

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Paweł Wójcik *Generalized orthogonality equation* (joint work with Tomasz Kania and Tomasz Kobos)

During this talk we show that every map between finite-dimensional normed spaces of the same dimension that respects fixed semi-inner products must be automatically a linear isometry. More precisely, the main results is as follows:

Theorem 1
Let X and Y be normed spaces with fixed semi-inner products $[\cdot|\cdot]_X$, $[\cdot|\cdot]_Y$, respectively. Suppose that $f\colon X \to Y$ is a function such that

$$[f(x)|f(y)]_Y = [x|y]_X, \qquad x, y \in X.$$

If either

(a) *X and Y have the same finite dimension, or*
(b) *X has a Schauder basis (e_i) and $(f(e_i))$ is a Schauder basis of Y, then f is a linear isometry.*

Moreover, we construct a uniformly smooth renorming of the Hilbert space ℓ_2 and a continuous injection acting thereon that respects the semi-inner products, yet it is non-linear.

Sebastian Wójcik *Quasi-arithmetic type means generated by the generalized Choquet integral*

We introduce a class of quasi-arithmetic type means generated by the generalized Choquet integral. They are related to the notion of certainty equivalent under Cumulative Prospect Theory and play a significant role, e.g. in a utility-based insurance contracts pricing (cf. [2]). We establish the characterizations of the equality, positive homogeneity, and translativity for such means. In particular, we generalize some results in [1].

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Amr Zakaria 🌍 *An invariance problem with six unknown functions* (joint work with Zsolt Páles)

Given two continuous functions $f, g\colon I \to \mathbb{R}$ such that g is nowhere zero on I and the ratio function f/g is strictly monotone on I, the *weighted two-variable Bajraktarević mean* $B_{f,g}\colon I^2 \times \mathbb{R}_+^2 \to I$ is defined by

$$B_{f,g}((x,y),(t,s)) := \Big(\frac{f}{g}\Big)^{-1}\Big(\frac{tf(x)+sf(y)}{tg(x)+sg(y)}\Big) \qquad x,y \in I,\ s,t \in \mathbb{R}_+.$$

The purpose of this talk is to solve the invariance of a Bajraktarević mean with respect to two weighted Bajraktarević means. In more details, we intend, under higher order regularity assumptions, to solve the functional equation

$$\frac{\varphi(B_{f,g}((x,y),(t,s))) + \varphi(B_{h,k}((x,y),(s,t)))}{\psi(B_{f,g}((x,y),(t,s))) + \psi(B_{h,k}((x,y),(s,t)))} = \frac{\varphi(x)+\varphi(y)}{\psi(x)+\psi(y)}, \qquad x,y \in I,$$

or equivalently,

$$\begin{vmatrix} \varphi\big(B_{f,g}((x,y),(t,s))\big) + \varphi\big(B_{h,k}((x,y),(s,t))\big) & \varphi(x)+\varphi(y) \\ \psi\big(B_{f,g}((x,y),(t,s))\big) + \psi\big(B_{h,k}((x,y),(s,t))\big) & \psi(x)+\psi(y) \end{vmatrix} = 0, \qquad x,y \in I,$$

where $f,g,h,k,\varphi,\psi\colon I \to \mathbb{R}$ are unknown continuous functions such that g,k,ψ are nowhere zero on I, the ratio functions f/g, h/k, φ/ψ are strictly monotone on I, and s,t are fixed different positive numbers. This invariance problem is more general than the functional equation investigated in [3].

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Marek Cezary Zdun *On a linarization of Koopmans recursion and its application in economics*

Let X be a metric space. Let $\succeq$ be a binary relation on infinite product X^∞ and $U\colon X^\infty \to \mathbb{R}$ be a continuous surjection, such that $(x_0,x_1,x_2,\ldots) \succeq (y_0,y_1,y_2,\ldots) \Leftrightarrow U(x_0,x_1,x_2,\ldots) \geq U(y_0,y_1,y_2,\ldots)$. This means that the relation $\succeq$ is represented by the utility function U. Here, the space X is treated as a set of consumption outcomes, $(x_0,x_1,x_2,\ldots) \in X^\infty$ as a sequence of outcomes consumed over time and the relation $\succeq$ describes the preference of choice of outcomes. We assume that U satisfies the Koopmans recursion $U(x_0,x_1,x_2,\ldots) = \varphi(x_0, U(x_1,x_2,\ldots))$, where $\varphi\colon X\times\mathbb{R} \to \mathbb{R}$ is a continuous function strictly increasing in its second variable. This is a fundamental axiom in the preference theory in economics (see [1]). Moreover, we assume that $\succeq$ satisfies the property of the preference called "impatience", that is for all $n \geq 1$, $a,b \in X^n$ and for all $(x_0,x_1,\ldots) \in$

X^∞, $(a,a,a,\dots) \succeq (b,b,b,\dots) \Leftrightarrow (a,b,x_0,x_1,\dots) \succeq (b,a,x_0,x_1,\dots)$ (see [2]). We consider the Koopmans problem, when the relation $\succeq$ can be represented by another utility function V satisfying the affine recursion $V(x_0,x_1,x_2,\dots) = \alpha(x_0)V(x_1,x_2,\dots)+\beta(x_0)$. We prove that this property holds if and only if there exists a homeomorphic solution F of the system of simultaneous linear functional equations $F(\varphi(x,t)) = \alpha(x)F(t)+\beta(x)$, $x \in X$, $t \in \mathbb{R}$ for some functions $\alpha, \beta\colon X \to \mathbb{R}$. If $\succeq$ satisfies impatience, then we give necessary and sufficient conditions that ensure its representation by an affine utility function.

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Weinian Zhang 🌍 *Iteration and Equations*

Iteration, repeating the same, is a simple operation, but it is important in the informatic era and difficult because of its nonlinearity. This talk surveys the recent advances on iteration and fractional iteration. Furthermore, the advances on various classes of iterative equations are introduced. We also discuss related topics on embeddability of discrete dynamical systems and input-to-state stability of continuous dynamical systems.

Thomas Zürcher *New solutions to the Matkowski-Wesołowski problem*

The Matkowski-Wesołowski problem asks for continuous monotone solutions $\varphi\colon [0,1] \to [0,1]$ of the equation

$$\varphi(x) = \varphi\Big(\frac{x}{2}\Big) + \varphi\Big(\frac{x+1}{2}\Big) - \varphi\Big(\frac{1}{2}\Big)$$

such that $\varphi(0) = 0$ and $\varphi(1) = 1$.

This talk is based on joint work with Janusz Morawiec, [1]. The goal is to present solutions of this problem that are not Hölder continuous.

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2. Problems and Remarks

1. Remark

In an abelian Polish group X denote by ($\mathcal{EHM}$, $\mathcal{SHM}$) $\mathcal{HM}$, $\mathcal{HT}$, $\mathcal{FN}$ the family of all (injectively, strongly) Haar-meager sets, Haar-thin sets, null-finite sets, respectively (see [1]).

After the talk titled *Remarks on classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$* Adam Ostaszewski asked what if we replace Borel hulls of those sets by closed hulls in the above notions.

Denote by $\sigma\overline{\mathcal{F}}$ σ-ideal generated by closed sets from $\mathcal{F} \subset 2^X$. From result in [1] and [2] we have

$$\sigma\overline{\mathcal{HT}} \subsetneq \sigma\overline{\mathcal{NF}} \subsetneq \sigma\overline{\mathcal{EHM}} = \sigma\overline{\mathcal{SHM}} \subsetneq \sigma\overline{\mathcal{HM}} \subsetneq \mathcal{HM}.$$

It is high probable that there is a Borel set $A \notin \sigma\overline{\mathcal{HM}}_{\mathbb{Z}^\omega}$ such that $A \notin \mathcal{C}_{\mathbb{Z}^\omega}$ (an example of a Borel meager group $H \notin \sigma\overline{\mathcal{HM}}_{\mathbb{Z}^\omega}$ from [1] can be useful). If we prove it consequently we obtain that sets from families

$$\mathcal{B} \setminus \sigma\overline{\mathcal{HM}},\ \mathcal{B} \setminus \sigma\overline{\mathcal{SHM}},\ \mathcal{B} \setminus \sigma\overline{\mathcal{EHM}},\ \mathcal{B} \setminus \sigma\overline{\mathcal{NF}},\ \mathcal{B} \setminus \sigma\overline{\mathcal{HT}}$$

generally needn't be in classes $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

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Eliza Jabłońska

2. Problem

Let $I \subseteq \mathbb{R}$ be a proper interval and $F\colon I \times I \to \mathbb{R}$ be an operation.

- *reflexive*, if $F(x,x) = x$ for every $x \in I$;
- *partially strictly increasing*, if the functions $x \mapsto F(x, y_0)$, $y \mapsto F(x_0, y)$ are strictly increasing for every fixed $x_0 \in I$ and $y_0 \in I$.
- *symmetric*, if $F(x,y) = F(y,x)$ for every $x, y \in I$;
- *bisymmetric*, if

$$F\big(F(x,y), F(u,v)\big) = F\big(F(x,u), F(y,v)\big) \tag{1}$$

for every $x, y, u, v \in I$.

Theorem 1 (Burai-K.-Szokol '21)
Every reflexive, partially strictly increasing, symmetric and bisymmetric operation $F\colon I^2 \to I$ is continuous.

Open problem 1
Is that true that every reflexive, partially strictly increasing and bisymmetric operation $F\colon I^2 \to I$ is automatically continuous?

Open problem 2
Is that true that every partially strictly increasing and bisymmetric operation $F\colon I^2 \to I$ is automatically continuous?

Let $G\colon I^n \to I$ be an n-ary operation. We say that G is

- *partially strictly increasing*, if every (one-dimensional) section is strictly increasing.
- *reflexive*, if $G(x, \dots, x) = x$ for all $x \in G$.
- *symmetric*, if $G(x_1, \dots, x_n) = G(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for all $x_1, \dots, x_n$ and $\sigma \in S_n$, where S_n is the permutation group of n elements.
- *bisymmetic*, if $G(G(x_{1,1}, \dots, x_{1,n}), G(x_{2,1}, \dots, x_{2,n}), \dots, G(x_{n,1}, \dots, x_{n,n})) = G(G(x_{1,1}, \dots, x_{n,1}), G(x_{1,2}, \dots, x_{n,2}), \dots, G(x_{1,n}, \dots, x_{n,n}))$ for all $x_{i,j} \in [a, b]$.

The following general questions arise naturally as n-ary analogue of the previous open problems.

Open problem 3
Let $G\colon I^n \to I$ be a partially strictly increasing and bisymmetric operation. Is that true that G is continuous?

Concrete analogue of our result.

Open problem 4
Let $G\colon I^n \to I$ be a reflexive, partially strictly increasing, symmetric and bisymmetric operation. Is that true that there is a proper interval $J \subset \mathbb{R}$ and a function operation $f\colon J \to I$, such that

$$G(x_1, \dots, x_n) = f\Big(\frac{f^{-1}(x_1) + \dots + f^{-1}(x_n)}{n}\Big), \qquad x_1, \dots, x_n \in I,$$

hence G is continuous?

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Gergely Kiss

3. Problem

Recall that a Banach space X is said to have the SVM *property*, provided that there is a constant $v < \infty$ such that for any set algebra $\mathcal{F}$ and any set function $\nu\colon \mathcal{F} \to X$ satisfying

$$\|\nu(A \cup B) - \nu(A) - \nu(B)\| \le 1 \qquad \text{for } A, B \in \mathcal{F},\ A \cap B = \varnothing$$

there exists a (finitely additive) vector measure $\mu\colon \mathcal{F} \to X$ such that $\|\nu(A) - \mu(A)\| \le v$ for each $A \in \mathcal{F}$. This notion was introduced and investigated in [3] with the main motivation arising from the beautiful Kalton-Roberts theorem [2] which says that the one-dimensional space $\mathbb{R}$ (and, consequently, all $\mathbb{R}^n$ as well as ℓ_∞) has the SVM property.

Most of the paper [3] was devoted to showing that many classical Banach spaces satisfy certain weaker forms of the SVM property, with some restrictions upon the cardinality of the set algebra $\mathcal{F}$ in question. However, for the SVM property itself, with no restrictions on $\mathcal{F}$, all examples produced in [3] were *injective* Banach spaces (a Banach space X is called injective if for every Banach space Y and every closed linear subspace Z of Y, each bounded linear operator $Z \to X$ admits an extension to a bounded linear operator $Y \to X$). It is known (see [1]) that the quotient space ℓ_∞/c_0 and the space $\ell_\infty^c(\omega_1)$ of all bounded countably supported sequences on the first uncountable ordinal ω_1, equipped with the supremum norm, are examples of *universally separably injective* spaces (i.e. they satisfy the above mentioned extension property, provided that the subspace Z is separable), yet they are not injective. This motivates the following question.

Problem 1 (see [3, Problem no. 2])
Do the Banach spaces ℓ_∞/c_0 and $\ell_\infty^c(\omega_1)$ satisfy the SVM property?

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Tomasz Kochanek

4. Problem

Recently, the following distance formula has been proved in [1]:

Theorem 1
Let $\mathbb{X} = \ell_p^2$, where $1 < p < \infty$. Let $\widetilde{x} = (a,b) \in \mathbb{X}$, let $(c,d) \neq (0,0)$ and let $\mathbb{Y} = span\{(c,d)\}$ be a one-dimensional subspace in $\mathbb{X}$ be such that $\widetilde{x} \notin \mathbb{Y}$. Then

$$dist(\widetilde{x}, \mathbb{Y}) = \frac{|ad - bc|}{(|c|^q + |d|^q)^{\frac{1}{q}}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The problem: Obtain analogous formulae for ℓ_p^n, where $n \geq 3$.

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Debmalya Sain

5. Problem

This is a shortened version of the problem presented during 51st ISFE meeting. In [1] it was shown that equation

$$h\Delta_h^2 f(x) = 0$$

is superstable. Therefore it is natural to ask if a similar result is true for the equation

$$h\Delta_h^n f(x) = 0$$

with $n \geq 3$. A positive solution of the problem could be used to obtain stability results for equations stemming from numerical integration.

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