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An extensive note on various fractional-order type operators and some of their effects to certain holomorphic functions

Abstract. The aim of this paper is to present background information in relation with some fractional-order type operators in the complex plane, which is designed by the fractional-order derivative operator(s). Next we state various implications of that operator and then we show some interesting-special results of those applications.

1. Related definitions, notations and notions

In the mathematical literature, operator theory and its applications have very important-fundamental roles. One of those extensive operators also is the fractional order operator of fractional calculus. It is often encountered with this operator as operators of both real-variable functions and complex-variable functions in the published literature. Especially, those mentioned operators frequently include in studies with an excessive number of theoretical-applied results.

Our paper is one of numerous applications of those operators described by fractional order calculus, which has been expressed as Tremblay operator (for certain functions with complex variable) in the recent passed years. For the details of those fractional-order (type) operators and some of their (comprehensive) applications, one may refer to the main works in [6], [21], [22], [23], [24] and [25], and also see the earlier papers in [2], [3], [8], [27], [11], [13], [14], [15], [16], [17], [18], [19] and [24] as numerous different investigations.

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Now let us present certain information consisting of various definitions, notations and also notions which will connected with those operators of fractional-order calculus. As a priority, let the notations $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$ denote the sets of the complex numbers, the real numbers and the natural numbers, respectively. Then, under the following conditions:

$$0 < \beta \leq 1, \quad 0 < \alpha \leq 1, \quad 0 \leq \alpha - \beta < 1 \quad \text{and} \quad 0 \leq \rho < 1, \tag{1}$$

for any function $\ell(z)$, which is holomorphic in $\mathbb{U} := \{z \in \mathbb{C} : \ |z| < 1\}$ and is of any form in the series expansion given by

$$\ell(z) := \mathcal{L}_{m,q}(z) := \hat{\alpha}_q z^q + \hat{\alpha}_{m+q} z^{m+q} + \hat{\alpha}_{m+q+1} z^{m+q+1} + \cdots \tag{2}$$

for every $m \in \mathbb{N}$, $q \in \mathbb{N}$, $\hat{\alpha}_q \in \mathbb{C} - \{0\}$ and $\hat{\alpha}_{n+q} \in \mathbb{C}$, the Tremblay operator (generally denoted by the notation $\mathcal{T}_{\alpha,\beta}\{\ell\}(z)$) is defined by

$$\mathcal{T}_{\alpha,\beta}\{\ell\}(z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} z^{1-\beta} \mathcal{D}_z^{\alpha-\beta}\{z^{\alpha-1}\ell\}(z), \tag{3}$$

where the notation $\mathcal{D}_z^{\rho}\{\ell\}(z)$ denotes the well-known fractional derivative operator (of order ρ) for the function $\ell(z)$ and it also defined by

$$\mathcal{D}_z^{\rho}\{\ell\}(z) = \frac{1}{\Gamma(1-\rho)} \frac{d}{dz} \int_0^z \frac{\ell(s)}{(z-s)^{\rho}} ds, \tag{4}$$

where $\ell(z)$ is a holomorphic function in any simply-connected region of any domain of the complex plane ($\mathbb{C}$) involving the origin and the multiplicity of the term $(z-s)^{-\rho}$ is insulated by making use of $\log(z-s)$ when $z - s > 0$. As basic references, one may refer to the works in [3], [6], [7], [10], [12], [13], [16] and [24], [25], [26], [28] and [29].

As it has been emphasized in the summary of this research note, this comprehensive investigation has two main purposes. The first requirement is to introduce the operators as a priority. This was done above. The second is to bring some effects for the related operators to the literature. Indeed, as one of numerous applications of the special operator (introduced in (3)) together with the fractional derivative operator (given in (4)) to any holomorphic function $\ell(z)$ being of any form as in (2), the first assertion, which is Lemma 1.1 below, can be easily demonstrated.

Moreover, by simple research, great number of important relationships between the extensive result (in Lemma 1.1) and the (confluent) hypergeometric functions can be also analyzed. At all events, since its proof and various series forms of its possible applications are so easy, their details are omitted here.

Nevertheless, for the main assertions, namely, Lemma 1.1 just below, and those indicated-possible-special forms (or relations), one can specially check the works given by the references in [1], [2], [3], [12], [13], [15] and [30].

Lemma 1.1
Under the conditions designated by (1)*, let z be any element of the domain $\mathbb{U}$ and also let the function $\ell(z)$ be of the form given in* (2)*. For this holomorphic function $\ell(z)$, the following assertion:*

$$\mathcal{T}_{\alpha,\beta}\{\ell\}(z) = \hat{\alpha}_q \frac{\Gamma(\beta)\Gamma(q+\alpha)}{\Gamma(\alpha)\Gamma(q+\beta)} z^q + \sum_{j=m+q}^{\infty} \hat{\alpha}_j \frac{\Gamma(\beta)\Gamma(j+\alpha)}{\Gamma(\alpha)\Gamma(j+\beta)} z^j \tag{5}$$

holds true for all z in $\mathbb{U}$.

The second assertion, which is Lemma 1.2 just below, is a well-known important tool for proving and then setting our main results. For its detail, one may refer to the main works given by [16] and [22].

Lemma 1.2
Let $\Phi(z)$ be a holomorphic function that is of the form specified by (2)*. For any point $w_0 \in \mathbb{U}$, if*

$$\max\{|\Phi(w)| : \ |w| \leq |w_0|, \ (w \in \mathbb{U})\} = |\Phi(w_0)|, \tag{6}$$

then there exists some (real) number λ such that

$$w_0 \Phi'(w_0) = \lambda \Phi(w_0), \tag{7}$$

where $\lambda \in \mathbb{R}$ with $\lambda \geq q$, $(q \in \mathbb{N})$.

2. Related results, conclusions and recommendations

In this section, a number of various applications of some operators generated by the fractional order derivative to the certain functions, which both are holomorph in $\mathbb{U}$ and have the complex-series expansion being of (2), are properly established. For its proof, it will be enough to take into consideration the well-known assertion, which is Lemma 1.2. Also, see the earlier results in the paper in [14], which may be consisted of considering from a different angle.

Theorem 2.1
Under the conditions of the parameters specified in (1) *and in consideration of the definitions of the fractional-order (type) operators introduced by* (3) *and* (4)*, let the parameters M, m, Δ and Ψ be such that $M \geq m$, $m \in \mathbb{N}$, $0 < \Delta \leq 1$ and $0 \leq \Psi < 2\pi$. For some $z \in \mathbb{U}$ and for any function $\ell(z)$ like the form in* (2)*, if*

$$\arg\left\{ z(\mathcal{T}_{\alpha,\beta}\{\ell\}(z))^{(q+1)} \right\} \neq \Psi + 2\pi M \tag{8}$$

holds, then

$$\left| (\mathcal{T}_{\alpha,\beta}\{\ell\}(z))^{(q)} - \hat{\alpha}_q I^{q,q}_{\alpha,\beta} \right| < \Delta \tag{9}$$

also holds, where

$$\left(\frac{d}{dz}\right)^q \{\mathcal{T}_{\alpha,\beta}\{\ell\}(z)\} =: (\mathcal{T}_{\alpha,\beta}\{\ell\}(z))^{(q)} \qquad (q \in \mathbb{N}) \tag{10}$$

and

$$I^{u,v}_{\alpha,\beta} = \frac{u!}{(u-v)!}\frac{\Gamma(\beta)\Gamma(u+\alpha)}{\Gamma(\alpha)\Gamma(v+\beta)} \qquad (v \leq u,\ u,v \in \mathbb{N}). \tag{11}$$

Proof. Let the function $\ell(z)$ be in the form (2). Under the conditions given by (1), there is need to define a suitable function for proving of the theorem above. Accordingly, in the light of the extensive information given by (5) (of Lemma 1.1) and (10) (together with (11)), we firstly define an implicit function $\mathcal{G}(z)$ as in the following definition consisting of the q-th derivative of the mentioned function $\ell(z)$ with respect to z,

$$(\mathcal{T}_{\alpha,\beta}\{\ell\}(z))^{(q)} = \hat{\alpha}_q I^{q,q}_{\alpha,\beta} + \Delta\mathcal{G}(z), \tag{12}$$

where $0 < \Delta \leq 1$, $\hat{\alpha}_q \in \mathbb{C} - \{0\}$ and $z \in \mathbb{U}$.

Clearly, the implicit function $\mathcal{G}(z)$, just presented by (12), is both holomorph in $\mathbb{U}$ and is of the form (2) when $q := m$, $m \in \mathbb{N}$. For this reason, of course, we can consider that function $\mathcal{G}(z)$ for the pending proof. Indeed, it follows from (12) that

$$z\frac{d}{dz}\left\{(\mathcal{T}_{\alpha,\beta}\{\ell\}(z))^{(q)}\right\} \equiv z(\mathcal{T}_{\alpha,\beta}\{\ell\}(z))^{(q+1)} = \Delta z\mathcal{G}'(z) \tag{13}$$

for all $q \in \mathbb{N}$ and for some $0 < \Delta \leq 1$ and $z \in \mathbb{U}$.

For now, we claim that the inequality $|\mathcal{G}(z)| < 1$ holds in the domain $\mathbb{U}$. In fact, in the opposite case, then, in accordance with the assertion (6) (of Lemma 1.2), there exists a point z_0 in $\mathbb{U}$ such that

$$\max\{|\mathcal{G}(z)| :\ |z| \leq |z_0|,\ z \in \mathbb{U}\} = |\mathcal{G}(z_0)| = \lambda,$$

which readily yields that

$$\mathcal{G}(z_0) = \lambda e^{i\phi} \qquad (\lambda > 0,\ 0 \leq \phi < 2\pi,\ z_0 \in \mathbb{U}).$$

Apart from this, the mentioned expression in (7) (of Lemma 1.2) also gives rise to

$$z_0\mathcal{G}'(z_0) = \kappa\mathcal{G}(z_0) = \kappa\lambda e^{i\phi} \qquad (\kappa \geq m,\ m \in \mathbb{N}).$$

Therefore, of course, for all $\kappa \geq m$, $m \in \mathbb{N}$, by putting $z := z_0$ and also by the help of the related assertions (above), the expression acquired in (13) instantly follows that the equivalent relations given by

$$\begin{aligned}\arg\left\{z_0(\mathcal{T}_{\alpha,\beta}\{\ell\}(z_0))^{(q+1)}\right\} &= \arg\{\Delta z_0\mathcal{G}'(z_0)\} = \arg\{\Delta\kappa\mathcal{G}(z_0)\}\\ &= \arg\{\Delta\kappa\lambda e^{i\phi}\} = \arg\{\kappa\lambda\Delta\} + \arg\{e^{i\phi}\}\\ &= \arg\{\kappa\} + \arg\{\lambda\} + \arg\{\Delta\} + \arg\{e^{i\phi}\}\\ &= 2\pi\tau + \phi,\end{aligned} \tag{14}$$

where $\lambda > 0$, $\kappa \geq m$ for $m \in \mathbb{N}$, $0 < \Delta \leq 1$ and $0 \leq \phi < 2\pi$. But, the result (14) evidently is inconsistently with the result in (8), when accepting the values of the preestablished parameters τ and ϕ as follows

$$\tau := M \quad \text{and} \quad \phi := \Psi.$$

Namely, there is no $z_0 \in \mathbb{U}$ satisfying the condition $|\mathcal{G}(z_0)| = \lambda$ for $\lambda > 0$. Hence, we can decide upon here that it has to be in the form satisfying $|\mathcal{G}(z)| < 1$ for all $z \in \mathbb{U}$. Consequently, for any holomorphic function $\mathcal{G}(z)$ like the form in (2), the well-organized expression as in (12) immediately follows that

$$\left|(\mathcal{T}_{\alpha,\beta}\{\ell\}(z_0))^{(q)} - \tilde{\alpha}_q \mathbb{I}^{q,q}_{\alpha,\beta}\right| = |\Delta \mathcal{G}(z)| = |\Delta||\mathcal{G}(z)| < \Delta$$

for $0 < \Delta \leq 1$ and $z \in \mathbb{U}$. Clearly, the inequality (just above) is equivalent to the inequality given by (9), which also is the provision of Theorem 2.1. Thus, this ends the desired proof.

So far, three comprehensive theorems have been established, which are the fundamental results of this paper. In relation with the main result, the parameters located in the both sections have got great importance. Of course, we will consider only the appropriate selections of the parameters. Namely, only three special results of the related theorem, as certain examples, will be considered. They are also associated with the multivalently holomorphic functions having the form in (2) and the normalized holomorphic functions being of the form in (2) when $q := 1 =: \hat{\alpha}$ there. For a variety of the mentioned forms of those special functions with their geometric applications and also some special relations between certain operators, one can spare time and effort for checking the main works in [6] and [26], look also over the special results in [3], [5], [9], [11], [20], [21], [23] and [26]. We leave it for the interested researchers to determine the possible others.

As the first implication can be deduced by Theorem 2.1, for $m := 1$ and $q := 2$, let

$$\hat{\alpha}_2 := 1, \quad \varphi(z) := \mathcal{L}_{1,2}(z) \quad \text{and} \quad \mathbf{T}_{\alpha,\beta}(z) := \mathcal{T}_{\alpha,\beta}\{\varphi\}(z).$$

Then, in consideration of (2) and (5) for some $z \in \mathbb{U}$, the following special definitions

$$\varphi(z) = z^2 + \hat{\alpha}_3 z^3 + \hat{\alpha}_4 z^4 + \cdots \tag{15}$$

and

$$\begin{aligned}\mathbf{T}_{\alpha,\beta}\{\varphi\}(z) :=& \frac{\alpha(1+\alpha)}{\beta(1+\beta)} z^2 + \hat{\alpha}_3 \frac{\alpha(1+\alpha)(2+\alpha)}{\beta(1+\beta)(2+\beta)} z^3 \\ &+ \hat{\alpha}_4 \frac{\alpha(1+\alpha)(2+\alpha)(3+\alpha)}{\beta(1+\beta)(1+\beta)(2+\beta)} z^4 + \cdots\end{aligned} \tag{16}$$

can be easily received, respectively. At the time, the following expressions, which possess the similar form composed in (10), when $q := 1$ there, given by

$$z\Big(\frac{d}{dz}\Big)^3 (\mathcal{T}_{\alpha,\beta}\{\varphi\}(z)) \equiv z\frac{d^3}{dz^3}(\mathcal{T}_{\alpha,\beta}\{\varphi\}(z)) \equiv z\frac{d^3}{dz^3}(\mathbf{T}_{\alpha,\beta}\{\varphi\}(z)) \tag{17}$$

and

$$z\Big(\frac{d}{dz}\Big)^2 (\mathcal{T}_{\alpha,\beta}\{\varphi\}(z)) \equiv z\frac{d^2}{dz^2}(\mathcal{T}_{\alpha,\beta}\{\varphi\}(z)) \equiv z\frac{d^2}{dz^2}(\mathbf{T}_{\alpha,\beta}\{\varphi\}(z)) \tag{18}$$

can be then promoted. Then, in light of the information between (15)-(18) and also by means of Theorem 2.1, the first proposition can be easily constituted given by Statement 2.1

Statement 2.1
For the admissible parameters in (1), let the function $\varphi(z)$ be in the form (15) and also let the operator $\mathbf{T}_{\alpha,\beta}\{\varphi\}(z)$ be in the form (16). Then, for some $z \in \mathbb{U}$ and also for the (2-valently) holomorphic function $\varphi(z)$, the following proposition holds:

$$\arg\left\{z\frac{d^3}{dz^3}(\mathbf{T}_{\alpha,\beta}\{\varphi\}(z))\right\} \neq \Psi + 2\pi M \Rightarrow \left|\frac{d^2}{dz^2}(\mathbf{T}_{\alpha,\beta}\{\varphi\}(z)) - 2\frac{\alpha(1+\alpha)}{\beta(1+\beta)}\right| < \Delta,$$

where M ≥ 1, $0 < \Delta \leq 1$ and $0 \leq \Psi < 2\pi$.

The second implication can be discovered by Theorem 2.1, for the values of $m := 1$ and $q := 1$, let

$$\hat{\alpha}_1 := 1, \quad \eta(z) := \mathcal{L}_{1,1}(z) \quad \text{and} \quad \mathbb{T}_{\alpha,\beta}(z) := \mathcal{T}_{\alpha,\beta}\{\eta\}(z).$$

Then, in consideration of (2) and (5), for some $z \in \mathbb{U}$, the following special definitions

$$\eta(z) = z + \hat{\alpha}_2 z^2 + \hat{\alpha}_3 z^3 + \cdots \tag{19}$$

and

$$\mathbb{T}_{\alpha,\beta}\{\eta\}(z) := \frac{\alpha}{\beta}z + \hat{\alpha}_2\frac{\alpha(1+\alpha)}{\beta(1+\beta)}z^2 + \hat{\alpha}_3\frac{\alpha(1+\alpha)(2+\alpha)}{\beta(1+\beta)(1+\beta)}z^3 + \cdots \tag{20}$$

can be easily specified, respectively. At that time, the terms, which have the similar form presented in (10) when $q := 1$ there, given by

$$z\left(\frac{d}{dz}\right)^2(\mathcal{T}_{\alpha,\beta}\{\eta\}(z)) \equiv z\frac{d^2}{dz^2}(\mathcal{T}_{\alpha,\beta}\{\eta\}(z)) \equiv z\frac{d^2}{dz^2}(\mathbb{T}_{\alpha,\beta}\{\eta\}(z)) \tag{21}$$

and

$$z\left(\frac{d}{dz}\right)^1(\mathcal{T}_{\alpha,\beta}\{\eta\}(z)) \equiv z\frac{d}{dz}(\mathcal{T}_{\alpha,\beta}\{\eta\}(z)) \equiv z\frac{d}{dz}(\mathbb{T}_{\alpha,\beta}\{\eta\}(z)) \tag{22}$$

can be then organized. Then, in light of the information between (19)-(22) and by the help of Theorem 2.1, the second proposition can be easily composed given by Statement 2.2.

Statement 2.2
For all admissible parameters in (1), let the function $\eta(z)$ be in the form (19) and also let the operator $\mathbb{T}_{\alpha,\beta}\{\eta\}(z)$ be in the form (20). Then, for the normalized holomorphic function $\eta(z)$, the following implication

$$\arg\left\{z\frac{d^2}{dz^2}(\mathbb{T}_{\alpha,\beta}\{\eta\}(z))\right\} \neq \Psi + 2\pi M \Rightarrow \left|\frac{d}{dz}(\mathbb{T}_{\alpha,\beta}\{\eta\}(z)) - \frac{\alpha}{\beta}\right| < \Delta,$$

is ensured, where $M \geq 1$, $0 < \Delta \leq 1$, $0 \leq \Psi < 2\pi$ and $z \in \mathbb{U}$.

The last implication of our main results, when taking the values α and β as $\alpha := \sigma =: \beta$, the following-equivalent-special relations:

$$\mathcal{T}_{\sigma,\sigma}\{\eta\}(z) \equiv \eta(z) \equiv \mathbb{T}_{\sigma,\sigma}\{\eta\}(z) \qquad (0 < \sigma \leq 1)$$

can be easily obtained as in [14]. Then, under these conditions (just above), Statement 2.2 gives us also the following special result (below), which is directly associated with some of geometric(-analytic) properties of any *normalized* holomorphic function like $\varsigma(z)$ having the form in (19). (Cf., e.g., [8]. See (also) [4] and [28].)

STATEMENT 2.3
Let the function $\eta(z)$ be in the form (19). Then, the following presuppositions

$$\arg(z\eta''(z)) \neq \Psi + 2\pi M \Rightarrow |\eta'(z) - 1| < \Delta \Rightarrow \Re e(\eta'(z)) > 1 - \Delta$$

are true, where $M \geq 1$, $0 < \Delta \leq 1$, $0 \leq \Psi < 2\pi$ and $z \in \mathbb{U}$.

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