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Compactness of quadruple band matrix operator and geometric properties

Abstract. In this work, we characterize the class of compact matrix operators from $c_0(Q)$, $c(Q)$ and $\ell_\infty(Q)$ into c_0, c and ℓ_∞, respectively, with the notion of the Hausdorff measure of noncompactness. Moreover, we determine some geometric properties of the sequence space $\ell_p(Q)$.

1. Introduction

A set denoted by w is a collection of all real (or complex) valued sequences and is a vector space according to point-wise addition and scalar multiplication. A sequence space is a vector subspace of w. The sets symbolized by ℓ_∞, c_0, c and ℓ_p are the spaces of all bounded, null, convergent and absolutely p-summable sequences, respectively, where $1 \leq p < \infty$.

A BK-space is a Banach sequence space X provided each of the maps $p_n \colon X \to \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for every $n \in \mathbb{N}$ (see [8]).

Given a BK-space $X \supset \phi$, if

$$\lim_{n\to\infty} \Big\| x - \sum_{k=0}^{n} x_k e^{(k)} \Big\|_X = 0 \qquad \text{for all } x \in X,$$

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then X is said to have AK-property, where ϕ is the set of all finite sequences and $e^{(k)}$ is a sequence whose only non-zero term is a 1 in the kth place for all $k \in \mathbb{N}$ (see [21]).

The sequence spaces ℓ_∞, c_0 and c are BK-spaces equipped with *sup-norm* defined by $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$ and ℓ_p is a BK-space according to *p-norm* defined by

$$\|x\|_p = \Big(\sum_{k=0}^{\infty} |x_k|^p\Big)^{\frac{1}{p}},$$

where $p \in [1, \infty)$.

Let X and Y be two arbitrary sequence spaces and $A = (a_{nk})$ be an infinite matrix of real (or complex) entries. The domain of A in the sequence space X is defined by

$$X_A = \{x = (x_k) \in w : \; Ax \in X\} \tag{1.1}$$

which is also a sequence space, where $Ax = ((Ax)_n)$ is called A-transform of x defined by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k$$

supposed to be convergent for all $n \in \mathbb{N}$. From now on, the summation without limits runs from 0 to ∞. The class of all matrices provided $X \subset Y_A$ is denoted by $(X : Y)$ and the sequence in the nth row of A is denoted by $A_n = \{a_{nk}\}_{k=0}^{\infty}$ for all $n \in \mathbb{N}$. If $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n, k \in \mathbb{N}$, then A is called a triangle which has a unique and triangle inverse A^{-1} (see [32]). Also, the transpose of A is denoted by A^t.

Define an infinite matrix $S = (s_{nk})$ named summation matrix such that

$$s_{nk} = \begin{cases} 1, & 0 \leq k \leq n, \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. Then, the spaces of all bounded and convergent series are defined by $bs = (\ell_\infty)_S$ and $cs = c_S$, respectively.

Given two normed space X and Y, $B(X : Y)$ stands for the set of all bounded linear operators on X into Y. The unit sphere and closed unit ball in X are defined by

$$S_X = \{x \in X : \; \|x\| = 1\}$$

and

$$B_X = \{x \in X : \; \|x\| \leq 1\}$$

respectively.

A linear operator L on X into Y is called compact if and only if for all bounded sequence (x_n) in X the sequence $(L(x_n))$ contains a convergent subsequence in Y. We denote the class of such operators by $C(X : Y)$ (see [21]).

For a given BK-space $X \supset \phi$ and a sequence $b = (b_k) \in w$, we write

$$\|b\|_X^* = \sup_{x \in B_X} \Big| \sum_{k=0}^{\infty} b_k x_k \Big|.$$

In the theory of summability, matrix transformations theory has a great importance. Cesàro, Norlund, Borel, Riesz and others have pioneered the development of the theory of summability. Many authors have used the matrix domain of difference matrices for constructing new sequence spaces and have investigated some properties of those spaces in $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ in [19], $\Delta c_0(p)$, $\Delta c(p)$ and $\Delta\ell_\infty(p)$ in [1], $c_0(u,\Delta,p)$, $c(u,\Delta,p)$ and $\ell_\infty(u,\Delta,p)$ in [2], $c_0(\Delta^2)$, $c(\Delta^2)$ and $\ell_\infty(\Delta^2)$ in [13], $c_0(u,\Delta^2)$, $c(u,\Delta^2)$ and $\ell_\infty(u,\Delta^2)$ in [26], $c_0(u,\Delta^2,p)$, $c(u,\Delta^2,p)$ and $\ell_\infty(u,\Delta^2,p)$ in [5], $c_0(\Delta^m)$, $c(\Delta^m)$ and $\ell_\infty(\Delta^m)$ in [14], $\hat{\ell_\infty}$, $\hat{c_0}$, $\hat{c}$ and $\hat{\ell_p}$ in [18], $c_0(B)$, $c(B)$, $\ell_\infty(B)$ and $\ell_p(B)$ in [29], $w_0^p(r,s)$, $w^p(r,s)$ and $w_\infty^p(r,s)$ in [10], $c_0(B)$, $\ell_\infty(B)$ and $\ell_p(B)$ in [9], $c_0(Q)$, $c(Q)$, $\ell_\infty(Q)$ and $\ell_p(Q)$ in [6], $f(Q(r,s,t,u))$, $f_0(Q(r,s,t,u))$ and $fs(Q(r,s,t,u))$ in [7]. Recently, studies on the matrix domain of generalized difference matrix Δ_i^3 have also been done some authors in [30], [31], [24], [25].

2. Compact operators on new sequence spaces derived by quadruple band matrix

In this part, we give some knowledge related to the sequence spaces $c_0(Q)$, $c(Q)$ and $\ell_\infty(Q)$ obtained from the domain of quadruple band matrix and characterize the class of compact matrix operators from $c_0(Q)$, $c(Q)$ and $\ell_\infty(Q)$ into c_0, c and ℓ_∞, respectively by means of the notion of the Hausdorff measure of noncompactness.

By using the domain of quadruple band matrix

$$Q = Q(r,s,t,u) = (q_{nk}(r,s,t,u)),$$

Bişgin defined the sequence spaces $c_0(Q)$, $c(Q)$ and $\ell_\infty(Q)$ in [6] as follows:

$$c_0(Q) = \big\{x=(x_k)\in w:\ \lim_{k\to\infty}(rx_k+sx_{k-1}+tx_{k-2}+ux_{k-3})=0\big\},$$

$$c(Q) = \big\{x=(x_k)\in w:\ \lim_{k\to\infty}(rx_k+sx_{k-1}+tx_{k-2}+ux_{k-3}) \text{ exists}\big\}$$

and

$$\ell_\infty(Q) = \big\{x=(x_k)\in w:\ \sup_{k\in\mathbb{N}}|rx_k+sx_{k-1}+tx_{k-2}+ux_{k-3}|<\infty\big\},$$

where $Q = Q(r,s,t,u) = (q_{nk}(r,s,t,u))$ is defined by

$$q_{nk}(r,s,t,u) = \begin{cases} r, & k=n,\\ s, & k=n-1,\\ t, & k=n-2,\\ u, & k=n-3,\\ 0, & \text{otherwise} \end{cases}$$

for all $n,k\in\mathbb{N}$ and $r,s,t,u\in\mathbb{R}\setminus\{0\}$.

Here, it is easy to check that $Q(1,-3,3,-1)=\Delta^3$, $Q(r,s,t,0)=B(r,s,t)$, $Q(1,-2,1,0)=\Delta^2$, $Q(r,s,0,0)=B(r,s)$ and $Q(1,-1)=\Delta$, where Δ^3, $B(r,s,t)$,

Δ^2, $B(r,s)$ and Δ are called third order difference, triple band, second order difference, double band (generalized difference) and difference matrix, respectively. Therefore, our results derived from the matrix domain of the quadruple band matrix are more general and more comprehensive than the results on the matrix domain of the others mentioned above.

Also, for a given $x=(x_k)\in w$, the Q-transform of x is written as follows:

$$(Qx)_k = y_k = rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3} \tag{2.1}$$

for all $k\in\mathbb{N}$.

Let us consider the equation

$$rz^3 + sz^2 + tz + u = 0$$

where $r,s,t,u\in\mathbb{R}\setminus\{0\}$. We know that this equation has three roots such that $z_1 = \frac{1}{3r}[a-b-s]$, $z_2 = -\frac{1}{6r}[(1-i\sqrt{3})a-(1+i\sqrt{3})b+2s]$ and $z_3 = -\frac{1}{6r}[(1+i\sqrt{3})a-(1-i\sqrt{3})b+2s]$, where

$$a = \sqrt[3]{\frac{\sqrt{(-27r^2u+9rst-2s^3)^2+4(3rt-s^2)^3}-27r^2u+9rst-2s^3}{2}}$$

and

$$b = \sqrt[3]{\frac{\sqrt{(-27r^2u+9rst-2s^3)^2+4(3rt-s^2)^3}+27r^2u-9rst+2s^3}{2}}$$

Here and in the following, unless stated otherwise, we assume that σ_1, σ_2 and σ_3 are random three roots of the equation $rz^3+sz^2+tz+u=0$.

Let $b=(b_k)\in w$. In this part, we suppose that the matrices $Q^{-1}=G=(g_{nk})$, $D=(d_{nk})$, $H^{(b)}=(h^{(b)}_{nk})$, $T=(t_{nk})$ and $V=(v_{nk})$ are defined by

$$g_{nk} = \begin{cases} \frac{1}{r}\sum\limits_{i=0}^{n-k}\sum\limits_{l=0}^{n-k-i}\sigma_1^{n-k-i-l}\sigma_2^l\sigma_3^i, & 0\le k\le n,\\ 0, & k>n,\end{cases}$$

$D=G^t$,

$$h^{(b)}_{nk} = \begin{cases} \frac{1}{r}\sum\limits_{j=n}^{\infty}\sum\limits_{i=0}^{j-k}\sum\limits_{l=0}^{j-k-i}\sigma_1^{j-k-i-l}\sigma_2^l\sigma_3^i b_j l, & 0\le k\le n,\\ 0, & k>n,\end{cases}$$

$T=DA_n$ and

$$v_{nk} = \frac{1}{r}\sum_{j=k}^{\infty}\sum_{i=0}^{j-k}\sum_{l=0}^{j-k-i}\sigma_1^{j-k-i-l}\sigma_2^l\sigma_3^i a_{nj}$$

for all $n,k\in\mathbb{N}$.

Lemma 2.1 (see [22], Theorem 3.2)

(a) *Given a BK-space X provided AK-property or $X = \ell_\infty$. Then, $b = (b_k) \in \{X_Q\}^\beta$ if and only if $b = (b_k) \in (X^\beta)_D$ and $H^{(b)} \in (X : c_0)$.*
In addition, if $b = (b_k) \in \{X_Q\}^\beta$, then we get

$$\sum_{k=0}^{\infty} b_k x_k = \sum_{k=0}^{\infty} (D_k a)(Q_k x)$$

for all $x \in X_Q$.

(b) *$b = (b_k) \in \{c_Q\}^\beta$ if and only if $b = (b_k) \in (\ell_1)_D$ and $H^{(b)} \in (c : c)$.*
In addition, if $b = (b_k) \in \{c_Q\}^\beta$, then we get

$$\sum_{k=0}^{\infty} b_k x_k = \sum_{k=0}^{\infty} (D_k a)(Q_k x) - \xi\alpha$$

for all $x \in c_Q$, where $\xi = \lim_{k\to\infty} Q_k x$ and $\alpha = \lim_{n\to\infty} \sum_{k=0}^{n} h_{nk}^{(b)}$.

Lemma 2.2 (see [22], Theorem 3.4)

(a) *Given an arbitrary subset Y of w and a BK-space X provided AK-property or $X = \ell_\infty$. Then $A \in (X_Q : Y)$ if and only if $T \in (X : Y)$ and $H^{(A_n)} \in (X : c_0)$ for all $n \in \mathbb{N}$.*
In addition, if $A \in (X_Q : Y)$, then we get

$$Ax = T(Qx)$$

for all $x \in X_Q$.

(b) *Given a linear subspace Y of w. Then, $A \in (c_Q : Y)$ if and only if*

$$T \in (c_0 : Y),\ H^{(A_n)} \in (c : c)$$

for all $n \in \mathbb{N}$ and

$$Te - (\alpha_n) \in Y,$$

where $\alpha_n = \lim_{m\to\infty} \sum_{k=0}^{m} h_{nk}^{(A_n)}$ for all $n \in \mathbb{N}$.
In addition, if $A \in (c_Q : Y)$, then we get

$$Ax = T(Qx) - \xi(\alpha_n)$$

for all $x \in c_Q$, where $\xi = \lim_{k\to\infty} Q_k x$.

Proposition 2.3 (see [12], Proposition 3.2)

(a) *If $X \in \{c_0,\ \ell_\infty\}$, then we have*

$$\|b\|_{X_Q}^{*} = \|Db\|_1 \tag{2.2}$$

for all $b = (b_k) \in \{X_Q\}^\beta$.

(b) *We have*

$$\|b\|^*_{c_Q} = \|Db\|_1 + |\alpha| \tag{2.3}$$

for all $b = (b_k) \in \{c_Q\}^\beta$, $\alpha = \lim_{n\to\infty} \sum_{k=0}^{n} h^{(b)}_{nk}$.

Definition 2.1 (see [23], Definition 2.10)
Given a metric space (X, d) and $E \in \mathcal{M}_X = \{E : E \subset X \text{ and } E \text{ bounded}\}$. Then, the Hausdorff measure of noncompactness of E is defined by

$$\chi(E) = \inf\left\{\epsilon > 0 : E \subset \bigcup_{i=1}^{n} B(x_i, \delta_i),\ x_i \in X,\ \delta_i < \epsilon\ (i = 1, 2, \ldots, n)\ n \in \mathbb{N}\right\},$$

where $B(x_i, \delta_i) = \{y \in X : d(x_i, y) < \delta_i\}$. Here, the function χ is called the Hausdorff measure of noncompactness.

Definition 2.2 (see [23], Definition 2.24)
Let χ_1 and χ_2 be the Hausdorff measure of noncompactness defined on the Banach spaces X and Y, respectively. The operator $L: X \to Y$ is called (χ_1, χ_2)-bounded if $L(E) \in \mathcal{M}_Y$ and there exists a positive constant k such that $\chi_2(L(E)) \le k\chi_1(E)$ for all $E \in \mathcal{M}_X$. If an operator L is (χ_1, χ_2)-bounded, the (χ_1, χ_2)-measure of noncompactness of L is defined by

$$\|L\|_{(\chi_1,\chi_2)} = \inf\{k > 0 : \chi_2(L(E)) \le k\chi_1(E) \text{ for every } E \in \mathcal{M}_X\}.$$

For brevity of notation, If $\chi_1 = \chi_2 = \chi$, then $\|L\|_{(\chi,\chi)} = \|L\|_\chi$.

Lemma 2.4
Given two BK-spaces X and Y.

(i) *For all $A \in (X : Y)$, there exists a $L_A \in B(X, Y)$ such that $L_A(x) = Ax (x \in X)$, namely $(X : Y) \subset B(X, Y)$ (see [23], Theorem 1.23).*

(ii) *If X has the property AK, then every $L \in B(X, Y)$ is given by a matrix $A \in (X : Y)$ such that $Ax = L(x)$ $(x \in X)$, namely $B(X, Y) \subset (X : Y)$ (see [17], Theorem 1.9).*

Lemma 2.5 (see [23], Theorem 1.23)
Given a BK-space X and $Y \in \{\ell_\infty, c_0, c\}$. Then, if $A \in (X : Y)$, we get

$$\|L_A\| = \|A\|_{(X,\infty)} = \sup_{n\in\mathbb{N}} \|A_n\|^*_X < \infty.$$

Lemma 2.6 (see [23], Theorem 2.25 and Corollary 2.26)
Given two Banach spaces X and Y and $L \in B(X, Y)$. Then, we write

$$\|L\|_\chi = \chi(L(B_X)) = \chi(L(S_X)), \tag{2.4}$$

$$\|L\|_\chi = 0 \text{ if and only if } L \in C(X, Y), \tag{2.5}$$

$$\|L\|_\chi \le \|L\|. \tag{2.6}$$

Lemma 2.7 (Goldenštein, Gohberg, Markus [23], Theorem 2.23)
Let X be a Banach space which has a Schauder basis $\{e_1, e_2, \ldots\}$, $E \in \mathcal{M}_X$ and $P_n\colon X \to X$ be the projector onto linear span of $\{e_1, e_2, \ldots\}$. Then, the following inequality holds.

$$\frac{1}{\gamma} \limsup_{n\to\infty} \Big(\sup_{x\in E} \|(I-P_n)(x)\| \Big) \le \chi(E) \le \limsup_{n\to\infty} \Big(\sup_{x\in E} \|(I-P_n)(x)\| \Big),$$

where $\gamma = \limsup_{n\to\infty} \|I - P_n\|$.

Lemma 2.8 (see [27], Theorem 2.8)
Given $X \in \{\ell_p, c_0\}$, where $1 \le p < \infty$ and $E \in \mathcal{M}_X$. For an operator $P_n\colon X \to X$ defined by $P_n(x) = x^{[n]}$ $(x \in X)$ the following statement holds

$$\chi(E) = \lim_{n\to\infty} \Big(\sup_{x\in E} \|(I-P_n)(x)\| \Big).$$

Theorem 2.9
Given $X \in \{c_0(Q), \ell_\infty(Q)\}$. Then the followings hold.

(i) *If $A \in (X : c_0)$, then*

$$\|L_A\|_\chi = \lim_{m\to\infty} \|V_{[m]}\|_{(X,\infty)}. \tag{2.7}$$

(ii) *If $A \in (X : \ell_\infty)$, then*

$$0 \le \|L_A\|_\chi \le \lim_{m\to\infty} \|V_{[m]}\|_{(X,\infty)}, \tag{2.8}$$

where $V_{[m]} = \sup_{n>m} \Big(\sum_{k=0}^{\infty} |v_{nk}| \Big)$ *for all* $m \in \mathbb{N}$.

Proof. In (2.7) and (2.8), the limits clearly exist.

(i) By applying Lemmas 2.6 and 2.8, we get

$$\|L\|_\chi = \chi(L(B_X)) = \lim_{m\to\infty} \Big(\sup_{x\in B_X} \|(I-P_m)(Ax)\| \Big), \tag{2.9}$$

where $P_m\colon c_0 \to c_0$, $P_m(x) = x^{[m]}$ for $x = (x_k) \in c_0$ and $m \in \mathbb{N}_0$. Now, let us define $A^{[m]} = (a^m_{nk})_{n,k=0}^\infty$ such that

$$a^m_{nk} = \begin{cases} 0, & 0 \le n \le m, \\ a_{nk}, & n > m \end{cases}$$

for all $n, k, m \in \mathbb{N}$. Then, because of $A^{[m]} \in (X : c_0)$ and so $A^{[m]}_n \in X^\beta$, by considering Lemmas 2.5 and 2.1 and (2.2) in Proposition 2.3, we get

$$\|A^{[m]}_n\|^*_X = \|DA^{[m]}_n\|_1 = \sum_{k=0}^\infty |D_k A^{[m]}_n| = \sum_{k=0}^\infty \sum_{j=k}^\infty g_{jk} a^m_{nj}.$$

By taking into account the definition $D = (d_{nk})$, we have

$$t_{nk} = D_k A_n = \frac{1}{r} \sum_{j=k}^{\infty} a_{nj} \sum_{i=0}^{j-k} \sum_{l=0}^{j-k-i} \sigma_1^{j-k-i-l} \sigma_2^l \sigma_3^i$$

for all $k \in \mathbb{N}_0$. Thus, we obtain

$$\|A_n^{[m]}\|_X^* = \sum_{k=0}^{\infty} |v_{nk}|$$

for $n > m$. As a result of these, we conclude that

$$\begin{aligned} \sup_{x \in B_X} \|(I - P_m)(Ax)\| = \|L_{A^{[m]}}\| = \sup_{n>m} \|A_n^{[m]}\|_X^* = \sup_{n>m} \left(\sum_{k=0}^{\infty} |v_{nk}| \right) \\ = \|V_{[m]}\|_{(X,\infty)}. \end{aligned} \tag{2.10}$$

So (2.7) follows from (2.9) and (2.10).

(ii) Now, let us define a projector $P_m \colon \ell_\infty \to \ell_\infty$, $P_m(x) = x^{[m]}$ for $x = (x_k) \in \ell_\infty$ and $m \in \mathbb{N}_0$. Then, by combining $L(B_X) \subset P_m(L(B_X)) + (I - P_m)(L(B_X))$, the known properties of χ(see [23], Theorem 2.12) and conditions (2.4) and (2.6), we write

$$\begin{aligned} \chi(L(B_X)) &\le \chi(P_m(L(B_X))) + \chi((I - P_m)(L(B_X))) \\ &= \chi((I - P_m)(L(B_X))) \le \sup_{x \in B_X} \|(I - P_m)(Ax)\| \\ &= \|L_{A^{[m]}}\|, \end{aligned}$$

which yields that the condition (2.8) holds. This completes the proof.

By connecting the Theorem 2.9 and the condition (2.5), we can give the following result.

Corollary 2.10
Given $X \in \{c_0(Q), \ell_\infty(Q)\}$.

(i) In case of $A \in (X : c_0)$, L_A is compact if and only if

$$\lim_{m\to\infty} \left[\sup_{n>m} \left(\sum_{k=0}^{\infty} |v_{nk}| \right) \right] = 0. \tag{2.11}$$

(ii) In case of $A \in (X : \ell_\infty)$, L_A is compact if the condition (2.11) holds.

Theorem 2.11
The following statements hold.

(i) *In case of* $A \in (c(Q) : c_0)$,

$$\|L_A\|_\chi = \lim_{m\to\infty} \|V_{[m]}\|_{(X,\infty)}.$$

(ii) *In case of* $A \in (c(Q) : \ell_\infty)$,

$$0 \leq \|L_A\|_\chi \leq \lim_{m\to\infty} \|V_{[m]}\|_{(X,\infty)},$$

where $\|V_{[m]}\|_{(X,\infty)} = \sup_{n>m} \Big(\sum_{k=0}^{\infty} |v_{nk}| + |\alpha_n| \Big)$ *and* $\alpha_n = \lim_{m\to\infty} \sum_{k=0}^{m} h_{mk}^{(A_n)}$.

Proof. In the proof of Theorem 2.9, if we change (2.2) with (2.3), theorem can be proved in a similar way.

By combining Theorem 2.11 and condition (2.5), the next corollary can be given.

Corollary 2.12

(i) In case of $A \in (c(Q) : c_0)$, L_A is compact if and only if

$$\lim_{m\to\infty} \Big[\sup_{n>m} \Big(\sum_{k=0}^{\infty} |v_{nk}| + |\alpha_n| \Big) \Big] = 0. \tag{2.12}$$

(ii) In case of $A \in (c(Q) : \ell_\infty)$, L_A is compact if the condition (2.12) holds.

Let us give two more results.

Proposition 2.13 (see [11], Corollary 5.13)
Define a sequence $\mu = (\mu_k)$ *such that* $\mu_k = \lim_{n\to\infty} t_{nk}$ *for all* $k \in \mathbb{N}_0$. *Given* $X \in \{\ell_\infty, c_0\}$. *Then,* $A \in (X_Q : c)$, *the following*

$$\frac{1}{2} \lim_{m\to\infty} \Big(\sup_{n\geq m} \|T_n - \mu\|_1 \Big) \leq \|L_A\|_\chi \leq \lim_{m\to\infty} \Big(\sup_{n\geq m} \|T_n - \mu\|_1 \Big)$$

holds.

Theorem 2.14 (see [11], Theorem 5.14)
Let $\xi = \lim_{n\to\infty} \Big(\sum_{k=0}^{\infty} t_{nk} - \alpha_n \Big)$ *and* $A \in (c_Q : c)$. *Then*

$$\frac{1}{2} B \leq \|L_A\|_\chi \leq B,$$

where

$$\lim_{m\to\infty} \Big[\sup_{n\geq m} \Big(\sum_{k=0}^{\infty} |t_{nk} - \mu_k| + \Big| \xi - \alpha_n - \sum_{k=0}^{\infty} \mu_k \Big| \Big) \Big].$$

Theorem 2.15
Define a sequence $\hat{\mu} = (\hat{\mu}_k)$ *such that* $\hat{\mu}_k = \lim_{n\to\infty} v_{nk}$ *for all* $k \in \mathbb{N}$ *and let* $\hat{\xi} = \lim_{n\to\infty} \Big(\sum_{k=0}^{\infty} v_{nk} - \alpha_n \Big)$.

(i) *Given* $X \in \{c_0(Q), \ell_\infty(Q)\}$, *then if* $A \in (X : c)$ *we have*

$$\frac{1}{2}B_1 = \lim_{m\to\infty}\Big[\sup_{n\geq m}\Big(\sum_{k=0}^{\infty}|v_{nk} - \hat{\mu}_k|\Big)\Big] \leq \|L_A\|_\chi = B_1,$$

where $B_1 = \lim_{m\to\infty}\|V_{[m]} - \hat{\mu}\|_{(X,\infty)}$.

(ii) *If* $A \in (c(Q) : c)$, *then*

$$\frac{1}{2}B_2 \leq \|L_A\|_\chi \leq B_3$$

where

$$B_2 = \lim_{m\to\infty}\Big[\sup_{n\geq m}\Big(\sum_{k=0}^{\infty}|v_{nk} - \hat{\mu}_k| + \Big|\hat{\xi} - \alpha_n - \sum_{k=0}^{\infty}\hat{\mu}_k\Big|\Big)\Big],$$

$$B_3 = \lim_{m\to\infty}\Big[\sup_{n\geq m}\Big(\sum_{k=0}^{\infty}|v_{nk} - \hat{\mu}_k| + \Big|\hat{\xi} - \lambda_n - \sum_{k=0}^{\infty}\hat{\mu}_k\Big|\Big)\Big].$$

Proof. By keeping in mind Proposition 2.13 and Theorem 2.14, one can see that the proof is obvious. Therefore, we omit it.

By combining Theorem 2.15 and condition (2.5), the next corollary can be given.

Corollary 2.16

(i) Given $X \in \{c_0(Q), \ell_\infty(Q)\}$. In case of $A \in (X : c)$, L_A is compact if and only if

$$\lim_{m\to\infty}\Big[\sup_{n\geq m}\Big(\sum_{k=0}^{\infty}|v_{nk} - \hat{\mu}_k|\Big)\Big] = 0.$$

(ii) In case of $A \in (c(Q) : c)$, L_A is compact if and only if

$$\lim_{m\to\infty}\Big[\sup_{n\geq m}\Big(\sum_{k=0}^{\infty}|v_{nk} - \hat{\mu}_k| + \Big|\hat{\xi} - \alpha_n - \sum_{k=0}^{\infty}\hat{\mu}_k\Big|\Big)\Big] = 0.$$

3. Geometric properties of the sequence space $\ell_p(Q)$

In this chapter, we determine some geometric properties of the space $\ell_p(Q)$. Given a Banach space $(X, \|.\|_X)$. If all bounded sequence $x = (x_n)$ contains a subsequence $y = (y_n)$ provided the Cesàro means $\frac{1}{n+1}\sum_{k=0}^{n} y_k$ are norm convergent, then X has the Banach-Saks property (see [4]).

If all weakly null sequence $x = (x_n)$ contains a subsequence $y = (y_n)$ provided the Cesàro means $\frac{1}{n+1}\sum_{k=0}^{n} y_k$ are norm convergent, then X has the weak Banach-Saks property (see [4]).

If all weakly null sequence $x = (x_n)$ has a subsequence $y = (y_n)$ provided

$$\Big\| \sum_{k=0}^{n} y_k \Big\|_X \leq M(n+1)^{\frac{1}{p}}$$

for some $M > 0$ and for all $n \in \mathbb{N}$, then X has Banach-Saks type p, where $1 < p < \infty$ (see [20]).

Given a weakly compact convex set $E \subset X$. Then, X is said to have the weak fixed point property, if every self mapping $T\colon E \to E$ that provides $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in E$ has a fixed point (see [15]).

For a given normed linear space X and a unit sphere $S(X)$ of X, the Gurarii's modulus of convexity is defined by

$$\beta_X(\epsilon) = \inf \big\{ 1 - \inf_{0 \leq \alpha \leq 1} \|\alpha x + (1-\alpha) y\| : \ x, y \in S(X), \ \|x - y\| = \epsilon \big\},$$

where $0 \leq \epsilon \leq 2$ (see [28]).

Theorem 3.1 (see [16])
A Banach space X has the weak fixed point property, if X provides the condition

$$R(X) = \sup \big\{ \liminf_{n \to \infty} \|x_n + x\| \big\} < 2,$$

where the supremum is taken over all weakly null sequences (x_n) of the unit ball and all points x of the unit ball.

Theorem 3.2
The sequence space $\ell_p(Q)$ is of the Banach-Saks type p.

Proof. Let us take a weakly null sequence $x = (x_n) \in B(\ell_p(Q))$ which is unit ball of $\ell_p(Q)$ and assume that (ϵ_n) is a sequence of positive numbers with $\sum \epsilon_n \leq \frac{1}{2}$. Define $y_0 = x_0 = 0$ and $y_1 = x_{n_1} = x_1$. Then, one can find an $m_1 \in \mathbb{N}$ such that

$$\Big\| \sum_{i=m_1+1}^{\infty} y_1(i) e^{(i)} \Big\|_{\ell_p(Q)} < \epsilon_1.$$

Resulting from $x_n \xrightarrow{w} 0$ implying $x_n \to 0$ coordinatewise, one can find an $n_2 \in \mathbb{N}$ such that

$$\Big\| \sum_{i=0}^{m_1} x_n(i) e^{(i)} \Big\|_{\ell_p(Q)} < \epsilon_1,$$

as $n \geq n_2$. Define $y_2 = x_{n_2}$. Then, one can find an $m_2 > m_1$ such that

$$\Big\| \sum_{i=m_2+1}^{\infty} y_2(i) e^{(i)} \Big\|_{\ell_p(Q)} < \epsilon_2.$$

By using $v_n \to 0$ coordinatewise once more, one can find an $n_3 > n_2$ such that

$$\Big\| \sum_{i=0}^{m_2} x_n(i) e^{(i)} \Big\|_{\ell_p(Q)} < \epsilon_2$$

as $n \geq n_3$. If this method is continued, one can derive two increasing sequences (m_k) and (n_k) such that

$$\Big\| \sum_{i=0}^{m_k} x_n(i) e^{(i)} \Big\|_{\ell_p(Q)} < \epsilon_k$$

for all $n \geq n_{k+1}$ and

$$\Big\| \sum_{i=m_k+1}^{\infty} y_2(i) e^{(i)} \Big\|_{\ell_p(Q)} < \epsilon_k,$$

where $y_k = x_{n_k}$. In this way

$$\begin{aligned}
&\Big\| \sum_{k=0}^{n} y_k \Big\|_{\ell_p(Q)} \\
&\quad = \Big\| \sum_{k=0}^{n} \Big(\sum_{i=0}^{m_{k-1}} y_k(i) e^{(i)} + \sum_{i=m_{k-1}+1}^{m_k} y_k(i) e^{(i)} + \sum_{i=m_k+1}^{\infty} y_k(i) e^{(i)} \Big) \Big\|_{\ell_p(Q)} \\
&\quad \leq \Big\| \sum_{k=0}^{n} \Big(\sum_{i=m_{k-1}+1}^{m_k} y_k(i) e^{(i)} \Big) \Big\|_{\ell_p(Q)} + 2 \sum_{k=0}^{n} \epsilon_k
\end{aligned}$$

and

$$\begin{aligned}
&\Big\| \sum_{k=0}^{n} \sum_{i=m_{k-1}+1}^{m_k} y_k(i) e^{(i)} \Big\|_{\ell_p(Q)}^{p} \\
&\quad = \sum_{k=0}^{n} \sum_{i=m_{k-1}+1}^{m_k} |r y_k(i) + s y_k(i-1) + t y_k(i-2) + u y_k(i-3)|^p \\
&\quad \leq \sum_{k=0}^{n} \sum_{i=0}^{\infty} |r y_k(i) + s y_k(i-1) + t y_k(i-2) + u y_k(i-3)|^p \leq n+1.
\end{aligned}$$

Therefore we get

$$\Big\| \sum_{k=0}^{n} y_k \Big\|_{\ell_p(Q)} \leq (n+1)^{\frac{1}{p}} + 1 \leq 2(n+1)^{\frac{1}{p}},$$

which yields that the space $\ell_p(Q)$ is of the Banach-Saks type p. This completes the proof.

In [6], Theorem 2.2 gives that $\ell_p(Q)$ is linearly isomorphic to ℓ_p. As a result of this, we write $R(\ell_p(Q)) = R(\ell_p) = 2^{\frac{1}{p}}$.

If we combine this result and Theorem 3.1, The next theorem can be given.

Theorem 3.3
The sequence space $\ell_p(Q)$ has the weak fixed point property, where $p \in [1, \infty)$.

Theorem 3.4
The Gurarii's modulus of convexity of the space $\ell_p(Q)$ holds

$$\beta_{\ell_p(Q)}(\epsilon) \leq 1 - \left[1 - \left(\frac{\epsilon}{2}\right)^p\right]^{\frac{1}{p}},$$

where $0 \leq \epsilon \leq 2$.

Proof. Given $0 \leq \epsilon \leq 2$. Let us suppose that the inverse of quadruple band matrix Q is G. Now, we define two sequences x and y by

$$x = \left(\left(G\left(1 - \left(\frac{\epsilon}{2}\right)^p\right)\right)^{\frac{1}{p}}, G\left(\frac{\epsilon}{2}\right), 0, 0, \dots\right),$$

$$y = \left(\left(G\left(1 - \left(\frac{\epsilon}{2}\right)^p\right)\right)^{\frac{1}{p}}, G\left(-\frac{\epsilon}{2}\right), 0, 0, \dots\right).$$

Then, we have

$$\|Qx\|_{\ell_p} = \|x\|_{\ell_p(Q)} = 1 \quad \text{and} \quad \|Qy\|_{\ell_p} = \|y\|_{\ell_p(Q)} = 1.$$

This gives us that $x, y \in S(\ell_p(Q))$ and $\|Qx - Qy\|_{\ell_p} = \|x - y\|_{\ell_p(Q)} = \epsilon$.

For given $0 \leq \alpha \leq 1$, we write

$$\|\alpha x + (1-\alpha)y\|^p_{\ell_p(Q)} = \|\alpha Qx + (1-\alpha)Qy\|^p_{\ell_p} = 1 - \left(\frac{\epsilon}{2}\right)^p + |2\alpha - 1|\left(\frac{\epsilon}{2}\right)^p$$

and

$$\inf_{0\leq\alpha\leq1} \|\alpha x + (1-\alpha)y\|^p_{\ell_p(Q)} = 1 - \left(\frac{\epsilon}{2}\right)^p \tag{3.1}$$

So, we have

$$\beta_{\ell_p(Q)}(\epsilon) \leq 1 - \left[1 - \left(\frac{\epsilon}{2}\right)^p\right]^{\frac{1}{p}}.$$

for $p \geq 1$. This completes the proof .

By considering the equality (3.1), the next results can be given.

Corollary 3.5
Because of $\beta_{\ell_p(Q)}(\epsilon) = 1$ for $\epsilon = 2$, the sequence space $\ell_p(Q)$ is strictly convex.

Corollary 3.6
Because of $0 < \beta_{\ell_p(Q)}(\epsilon) \leq 1$ for $0 < \epsilon \leq 2$, the sequence space $\ell_p(Q)$ is uniformly convex.

4. Conclusion

By taking into account the definition of Quadruple band matrix, we can derive that $Q(1, -3, 3, -1) = \Delta^3$, $Q(r, s, t, 0) = B(r, s, t)$, $Q(1, -2, 1, 0) = \Delta^2$, $Q(r, s, 0, 0) = B(r, s)$ and $Q(1, -1) = \Delta$, where Δ^3, $B(r, s, t)$, Δ^2, $B(r, s)$ and Δ are called third order difference, triple band, second order difference, double band (generalized difference) and difference matrix, respectively. Moreover, Quadruple band matrix is not a special case of m-th order generalized difference matrix B^m defined in [3] and is not a special case of the weighed mean matrices. Thus, our work fills up a gap in the known literature.

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