sciendo

Ann. Univ. Paedagog. Crac. Stud. Math. 21 (2022), 33-42
DOI: 10.2478/aupcsm-2022-0003

FOLIA 355

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica 21 (2022)

Takashi Noiri and Bishwambhar Roy[1]

$\gamma_\mu\mathcal{H}$-compactness in GTS

Abstract. Using the notion of operations on a generalized topological space (X, μ) and a hereditary class we have introduced the notion of γ_μ-compactness modulo a hereditary class $\mathcal{H}$ termed as $\gamma_\mu\mathcal{H}$-compactness. We have studied $\gamma_\mu\mathcal{H}$-compact spaces and $\gamma_\mu\mathcal{H}$-compact sets relative to μ.

1. Introduction

In 1979, Kasahara [7] introduced the notion of an operation on a topological space and introduced the concept of α-closed graph of a function. After then Janković [6] defined the concept of α-closed sets and investigated some properties of functions with α-closed graphs. In 1991, Ogata [10] introduced the notion of γ-open sets to investigate some new separation axioms of a topological space. Recently, Krishnan et al. [8] and Van An et al. [18] investigated the notion of operations on the family of all semi-open sets and pre-open sets. The notion of compactness is one of the most important area of research in mathematics.

The concept of $\mathcal{I}$-compactness was introduced in [9], while the notion of γ-compact spaces was studied in [2]. Recently, Carpintero et al. [1] introduced the concept of μ-compactness with respect to a hereditary class $\mathcal{H}$. The notion of weakly $\mu\mathcal{H}$-compact spaces was studied in [11, 12].

In this paper, by using hereditary classes [19, 5], a generalized topology [3] and an operation [15], we define the notion of γ_μ-compactness modulo a hereditary class called $\gamma_\mu\mathcal{H}$-compactness. We have studied several properties of $\gamma_\mu\mathcal{H}$-compact

AMS (2020) Subject Classification: 54C10, 54C08, 54D10.
Keywords and phrases: μ-open set, operation, Hereditary class, $\gamma_\mu\mathcal{H}$-compact.
[1] The author acknowledges the financial support from Department of Higher education, Science and technology and Biotechnology, Government of West Bengal, INDIA
ISSN: 2081-545X, e-ISSN: 2300-133X.

spaces and $\gamma_\mu\mathcal{H}$-compact sets. Finally, we have given some preservation theorems of $\gamma_\mu\mathcal{H}$-compact sets.

We recall some notions defined in [3]. Let X be a non-empty set and $exp\, X$ be the power set of X. We call a class $\mu \subsetneqq exp\, X$ a generalized topology [3], (briefly, GT) if $\emptyset \in \mu$ and the union of any elements of μ belongs to μ. A set X with a GT μ on it is called a *generalized topological space* (briefly, GTS) and is denoted by (X, μ).

For a GTS (X, μ), the elements of μ are called μ-open sets and the complement of μ-open sets are called μ-closed sets. For $A \subsetneqq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e. the smallest μ-closed set containing A; and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e. the largest μ-open set contained in A (see [3, 4]). It is easy to observe that i_μ and c_μ are idempotent and monotonic, where $\gamma\colon exp\, X \to exp\, X$ is said to be idempotent iff for each $A \subsetneqq X$, $\gamma(\gamma(A)) = \gamma(A)$, and monotonic iff $\gamma(A) \subsetneqq \gamma(B)$, whenever $A \subsetneqq B \subsetneqq X$. It is also well known from [4] that let μ be a GT on X, $A \subsetneqq X$ and $x \in X$, then $x \in c_\mu(A)$ if and only if $x \in M \in \mu$ implies $M \cap A \neq \emptyset$ and that $c_\mu(X \setminus A) = X \setminus i_\mu(A)$.

We recall that a collection $\mathcal{H}$ of subsets of X is a hereditary class (see [5]) if it is closed under subsets, i.e. if $A \in \mathcal{H}$ and $B \subsetneqq A$ then $B \in \mathcal{H}$.

2. $\gamma_\mu\mathcal{H}$-compact subsets

Definition 2.1 ([15])
Let (X, μ) be a GTS. An operation γ_μ on a generalized topology μ is a mapping from μ to $\mathcal{P}(X)$ (where $\mathcal{P}(X)$ is the power set of X) with $G \subsetneqq G^{\gamma_\mu}$ for each $G \in \mu$. This operation is denoted by $\gamma_\mu\colon \mu \to \mathcal{P}(X)$. Note that $\gamma_\mu(A)$ and A^{γ_μ} are two different notations for the same set.

Definition 2.2 ([15])
Let (X, μ) be a GTS and γ_μ an operation on μ. A subset G of a GTS (X, μ) is said to be γ_μ-open if for each point x of G, there exists a μ-open set U containing x such that $U^{\gamma_\mu} \subsetneqq G$.

A subset of a GTS (X, μ) is said to be γ_μ-closed if its complement is γ_μ-open in (X, μ). We shall use the symbol $\gamma_\mu O(X)$ to mean the collection of all γ_μ-open sets of the GTS (X, μ).

Remark 2.1 ([15])
Let (X, μ) be a GTS and $\gamma_\mu\colon \mu \to \mathcal{P}(X)$ be an operation on μ. We note that $\gamma_\mu O(X)$ is a GT on X. We also observe that every γ_μ-open set is a μ-open set, i.e. $\gamma_\mu O(X) \subsetneqq \mu$.

Definition 2.3 ([1])
Let (X, μ) be a GTS and $\mathcal{H}$ be a hereditary class on X. A subset $A \subsetneqq X$ is called $\mu\mathcal{H}$-compact if for every cover $\{U_\alpha :\ \alpha \in \Lambda\}$ of A by μ-open sets of X, there exists a finite subcollection $\{U_\alpha :\ \alpha \in \Lambda_0\}$ such that $A \setminus \bigcup\{U_\alpha :\ \alpha \in \Lambda_0\} \in \mathcal{H}$. X is called a $\mu\mathcal{H}$-compact space if X is $\mathcal{H}$-compact as a subset.

Definition 2.4 ([16])
Let (X,μ) be a GTS and $\gamma_\mu\colon \mu \to \mathcal{P}(X)$ be an operation on μ. A subset A of (X,μ) is said be γ_μ-compact relative to μ if for every cover $\{U_\alpha : \alpha \in \Lambda\}$ of A by μ-open subsets of X, there exists a finite subset Λ_0 of Λ such that $A \subseteq \bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\}$. If $A = X$, then X is called a γ_μ-compact space.

Definition 2.5
Let (X,μ) be a GTS, $\gamma_\mu\colon \mu \to \mathcal{P}(X)$ an operation on μ and $\mathcal{H}$ a hereditary class on X. A subset A of X is said to be $\gamma_\mu\mathcal{H}$-compact relative to μ if for every cover $\{U_\alpha : \alpha \in \Lambda\}$ of A by μ-open sets of X, there exists a finite subset Λ_0 of Λ such that $A\setminus\bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\} \in \mathcal{H}$. X is said to be $\gamma_\mu\mathcal{H}$-compact if X is $\gamma_\mu\mathcal{H}$-compact relative to μ.

Remark 2.2
(1) Let A be $\gamma_\mu\mathcal{H}$-compact relative to μ.
  (i) If $\mathcal{H} = \{\emptyset\}$, then A is γ_μ-compact (see [16]).
  (ii) If $\mathcal{H} = \{\emptyset\}$ and γ_μ is identity, then A is μ-compact (see [14]).
  (iii) If $\mathcal{H} = \{\emptyset\}$ and γ_μ is c_μ, then A is weakly μ-compact (see [17]).

(2) Let (X,μ) be $\gamma_\mu\mathcal{H}$-compact.
  (i) If γ_μ is identity, then X is $\mu\mathcal{H}$-compact (see [1]).
  (ii) If γ_μ is c_μ, then X is weakly $\mu\mathcal{H}$-compact (see [11]).

In fact, we can recover almost all the well known classical concepts of compactness by replacing the generalized topologies, operations and the hereditary classes.

Observation 2.3
Since $U \subseteq U^{\gamma_\mu}$ for every μ-open set $U \in \mu$, it follows that every $\mu\mathcal{H}$-compact space is $\gamma_\mu\mathcal{H}$-compact. But the converse is not true as shown by the next example. We also note that every γ_μ-compact subset is $\gamma_\mu\mathcal{H}$-compact but the converse is not true.

Remark 2.4
(a) Let $\mathbb{R}$ be the set of real numbers. Let $\mu = \{A \subseteq \mathbb{R} : A \text{ is infinite}\}$. Then $(\mathbb{R},\mu)$ is a GTS. If we take $\mathcal{H} = \{A \subseteq \mathbb{R} : A \text{ is finite}\}$, then $\mathcal{H}$ is a hereditary class on $\mathbb{R}$. It is easy to check that $(\mathbb{R},\mu)$ is not $\mu\mathcal{H}$-compact. In fact, $\{(-n,n) : n \in \mathbb{N}\}$ is a cover of $\mathbb{R}$ by μ-open sets of $\mathbb{R}$, but there do not exist any $r \in \mathbb{N}$ such that $\mathbb{R}\setminus\bigcup\{(-n,n) : n = 1,2,\ldots,r\} \in \mathcal{H}$. Let $\gamma_\mu\colon \mu \to \mathcal{P}(X)$ be an operation defined by $A^{\gamma_\mu} = A$, if $1 \notin A$ and $A^{\gamma_\mu} = \mathbb{R}$, if $1 \in A$. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of $\mathbb{R}$ by μ-open sets of $\mathbb{R}$. Then there exists U_{α_0} such that $1 \in U_{\alpha_0}$. Hence $\mathbb{R}\setminus U_{\alpha_0}^{\gamma_\mu} = \emptyset \in \mathcal{H}$. This shows that $\mathbb{R}$ is $\gamma_\mu\mathcal{H}$-compact relative to μ.

(b) Let $\mathbb{R}$ be the set of real numbers and $\mu = \tau_{co}$ (countable topology on $\mathbb{R}$). Let $\mathcal{H} = \{A \subseteq \mathbb{R} : m(A) = 0\}$ ($m(A)$ = the measure of A). Consider $\gamma_\mu\colon \mu \to \mathcal{P}(X)$ as the identity operation on μ. It is easy to check that $\mathbb{R}$ is not γ_μ-compact relative to μ but $\gamma_\mu\mathcal{H}$-compact relative to μ.

Theorem 2.5
Let (X,μ) be a GTS, $\gamma_\mu\colon \mu \to \mathcal{P}(X)$ an operation on μ and $\mathcal{H}$ an ideal on X. Then the union of two $\gamma_\mu\mathcal{H}$-compact subsets relative to μ is also a $\gamma_\mu\mathcal{H}$-compact set relative to μ.

Proof. Let A and B be $\gamma_\mu\mathcal{H}$-compact relative to μ. Let $\{U_\alpha :\ \alpha \in \Lambda\}$ be any cover of $A \cup B$ by μ-open subsets of X. Then there exist finite subsets Λ_1 and Λ_2 of Λ such that $A \setminus \bigcup\{U_\alpha^{\gamma_\mu} :\ \alpha \in \Lambda_1\} \in \mathcal{H}$ and $B \setminus \bigcup\{U_\alpha^{\gamma_\mu} :\ \alpha \in \Lambda_2\} \in \mathcal{H}$. Since $\Lambda_1 \cup \Lambda_2$ is a finite subset of Λ and $\mathcal{H}$ is an ideal, $A \cup B \setminus \bigcup\{U_\alpha^{\gamma_\mu} :\ \alpha \in \Lambda_1 \cup \Lambda_2\} \subseteqq A \setminus \bigcup\{U_\alpha^{\gamma_\mu} :\ \alpha \in \Lambda_1\} \cup B \setminus \bigcup\{U_\alpha^{\gamma_\mu} :\ \alpha \in \Lambda_2\} \in \mathcal{H}$. Thus $A \cup B$ is $\gamma_\mu\mathcal{H}$-compact relative to μ.

The following example shows that if the class $\mathcal{H}$ is not an ideal, then the union of two $\gamma_\mu\mathcal{H}$-compact subsets relative to μ is not necessarily $\gamma_\mu\mathcal{H}$-compact relative to μ.

Example 2.6
Let $X = (0,1)$ and μ be the restriction of the usual topology on X and $\mathcal{H} = \big\{A : A \subseteqq \big(0, \frac{1}{2}\big) \text{ or } A \subseteqq \big(\frac{1}{2}, 1\big)\big\}$. Let $\gamma_\mu\colon \mu \to \mathcal{P}(X)$ be the identity function. Then $A = \big(0, \frac{1}{2}\big)$ and $B = \big(\frac{1}{2}, 1\big)$ are $\gamma_\mu\mathcal{H}$-compact relative to μ. But their union $A \cup B$ is not so. In fact, $\big\{\big(\frac{1}{n}, 1 - \frac{1}{n}\big) :\ n \in \mathbb{N}\big\}$ is a cover of $A \cup B$ by μ-open sets. But there exist no $n_1, n_2, \ldots, n_k$ such that $A \cup B \setminus \bigcup\big\{\big(\frac{1}{n_i}, 1 - \frac{1}{n_i}\big) :\ i = 1, 2, \ldots, k\big\} \notin \mathcal{H}$.

Definition 2.6 ([16])
A GTS (X,μ) is called a γ_μ-regular space if for each point x of X and each μ-open set V containing x, there exists a μ-open set U containing x such that $\gamma_\mu(U) \subseteqq V$.

Theorem 2.7 ([16])
For a strong GTS (X,μ), the following properties are equivalent:

(i) $\mu = \gamma_\mu\text{-}O(X)$.

(ii) (X,μ) *is γ_μ-regular.*

(iii) *For each $x \in X$ and each $U \in \mu$ containing x, there exists $W \in \gamma_\mu\text{-}O(X)$ such that $x \in W \subseteqq W^{\gamma_\mu} \subseteqq U$.*

Theorem 2.8
Let (X,μ) be a GTS, where $\gamma_\mu\colon \mu \to \mathcal{P}(X)$ be an operation on μ. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) *hold. However, if (X,μ) is γ_μ-regular, then for any subset $A \subseteqq X$ the following are equivalent:*

(i) *A is $\mu\mathcal{H}$-compact relative to μ.*

(ii) *A is $\gamma_\mu\mathcal{H}$-compact relative to μ.*

(iii) *A is $\mu\mathcal{H}$-compact relative to $\gamma_\mu O(X)$.*

(iv) *A is $\gamma_\mu\mathcal{H}$-compact relative to $\gamma_\mu O(X)$.*

Proof. (i) $\Rightarrow$ (ii): It follows from the fact that for every μ-open set, $U \subseteqq U^{\gamma_\mu}$.

(ii) $\Rightarrow$ (iii): Let A be $\gamma_\mu\mathcal{H}$-compact relative to μ and $\{U_\alpha :\ \alpha \in \Lambda\}$ be a cover of A by γ_μ-open sets in X. Then for each $x \in A$, there exists $\alpha(x) \in \Lambda$ such that

$x \in U_{\alpha(x)}$. Since $U_{\alpha(x)}$ is γ_μ-open, there exists $V_{\alpha(x)} \in \mu$ such that $x \in V_{\alpha(x)} \subseteqq V_{\alpha(x)}^{\gamma_\mu} \subseteqq U_{\alpha(x)}$. Now $\{V_{\alpha(x)} : x \in A\}$ is a cover of A by μ-open sets of X. Then by (ii), there exists a finite subset A_0 of A such that $A \setminus \bigcup\{V_{\alpha(x)}^{\gamma_\mu} : x \in A_0\} \in \mathcal{H}$. Thus, $A \setminus \bigcup\{U_{\alpha(x)} : x \in A_0\} \in \mathcal{H}$. This shows that A is $\mu\mathcal{H}$-compact relative to $\gamma_\mu O(X)$.

(iii) $\Rightarrow$ (iv): The proof follows from the similar argument as in (i) $\Rightarrow$ (ii).

(iv) $\Rightarrow$ (i): Let $A \subseteqq X$ be $\gamma_\mu\mathcal{H}$-compact relative to $\gamma_\mu O(X)$. We shall show that A is $\mu\mathcal{H}$-compact relative to μ. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of A by μ-open sets of X. Then for each $x \in A$, there exists $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)}$. Since (X, μ) is γ_μ-regular, there exists $V_{\alpha(x)} \in \mu$ such that $x \in V_{\alpha(x)} \subseteqq V_{\alpha(x)}^{\gamma_\mu} \subseteqq U_{\alpha(x)}$. Since $\{V_{\alpha(x)} : x \in A\}$ is a cover of A by μ-open sets of X and A is $\gamma_\mu\mathcal{H}$-compact relative to $\gamma_\mu O(X)$, there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of A such that $A \setminus \bigcup\{V_{\alpha_{(x_i)}}^{\gamma_\mu} : i = 1, 2, \ldots, n\} \in \mathcal{H}$. Thus $A \setminus \bigcup\{U_{\alpha_{(x_i)}} : i = 1, 2, \ldots, n\} \in \mathcal{H}$. Hence A is $\mu\mathcal{H}$-compact relative to μ.

Corollary 2.9

Let (X, μ) be a GTS, where $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation on μ. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) hold. However, if (X, μ) is γ_μ-regular, then the following are equivalent:

(i) X is $\mu\mathcal{H}$-compact relative to μ.

(ii) X is $\gamma_\mu\mathcal{H}$-compact relative to μ.

(iii) X is $\mu\mathcal{H}$-compact relative to $\gamma_\mu O(X)$.

(iv) X is $\gamma_\mu\mathcal{H}$-compact relative to $\gamma_\mu O(X)$.

Example 2.10

Let $\mathbb{R}$ be the set of real numbers and $\mu = \{A \subseteqq \mathbb{R} : A \text{ is infinite}\} \cup \{\emptyset\}$. Then $(\mathbb{R}, \mu)$ is a GTS. Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be defined by $A^{\gamma_\mu} = \emptyset$ if $A = \{\emptyset\}$ and $\mathbb{R}$, otherwise. If we take $\mathcal{H} = \{A : A \text{ is finite}\}$, then $\mathcal{H}$ is a hereditary class on $\mathbb{R}$. It is easy to check that $\gamma_\mu O(\mathbb{R}) = \{\emptyset, \mathbb{R}\}$. Hence, $\mathbb{R}$ is $\gamma_\mu\mathcal{H}$-compact relative to $\gamma_\mu O(\mathbb{R})$, but not $\gamma_\mu\mathcal{H}$-compact relative to μ.

Theorem 2.11

Let (X, μ) be a GTS, $\gamma_\mu : \mu \to \mathcal{P}(X)$ an operation on μ and $\mathcal{H}_f$ be the hereditary class of finite subsets on X. Then $A \subseteqq X$ is γ_μ-compact relative to μ if and only if A is $\gamma_\mu\mathcal{H}_f$-compact relative to μ.

Proof. If A is γ_μ-compact relative to μ, then it is clearly $\gamma_\mu H_f$-compact relative to μ. Next, let A be $\gamma_\mu H_f$-compact relative to μ. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a μ-open cover of A. Then there exists a finite subset Λ_0 of Λ such that $A \setminus \bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\} \in \mathcal{H}_f$. Let $A \setminus \bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\} = \{x_1, x_2, \ldots, x_n\}$. For each $1 \leq j \leq n$, choose U_{α_j} such that $x_j \in U_{\alpha_j} \subseteqq U_{\alpha_j}^{\gamma_\mu}$. Thus $A \subseteqq \bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\} \cup \{U_{\alpha_j}^{\gamma_\mu} : j = 1, 2, \ldots, n\}$. Hence A is γ_μ-compact relative to μ.

Theorem 2.12

Let (X, μ) be a GTS, $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation on μ and $\mathcal{H}$ be a hereditary class on X. If A is $\gamma_\mu\mathcal{H}$-compact relative to μ and B is γ_μ-closed, then $A \cap B$ is $\gamma_\mu\mathcal{H}$-compact relative to μ.

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of $A \cap B$ by μ-open subsets of X. Then $X \setminus B$ is γ_μ-open and $A \setminus B \subseteq X \setminus B$. Then for each $x \in A \setminus B$, there exists a μ-open set U_x containing x such that $x \in U_x \subseteq U_x^{\gamma_\mu} \subseteq X \setminus B$. Then $\{U_\alpha : \alpha \in \Lambda\} \cup \{U_x : x \in A \setminus B\}$ is a cover of A by μ-open sets in X. So there exist finite subsets Λ_0 of Λ and A_0 of A such that $A \subseteq \big[\bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\}\big] \cup \big[\bigcup\{U_x^{\gamma_\mu} : x \in A_0\}\big] \cup H$ for some $H \in \mathcal{H}$. Then $A \cap B \subseteq \big[\bigcup\{U_\alpha^{\gamma_\mu} \cap B : \alpha \in \Lambda_0\}\big] \cup \big[\bigcup\{U_x^{\gamma_\mu} \cap B : x \in A_0\}\big] \cup (H \cap B) \subseteq \bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\} \cup H$. Thus $A \cap B \setminus \bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\} \subseteq H \in \mathcal{H}$. Hence $A \cap B$ is $\gamma_\mu\mathcal{H}$-compact relative to μ.

Corollary 2.13
Let (X, μ) be a GTS, $\gamma_\mu \colon \mu \to \mathcal{P}(X)$ be an operation on μ and $\mathcal{H}$ be a hereditary class on X. If X is $\gamma_\mu\mathcal{H}$-compact and B is γ_μ-closed, then B is $\gamma_\mu\mathcal{H}$-compact relative to μ.

Definition 2.7
Let (X, μ) be a GTS, $\gamma_\mu \colon \mu \to \mathcal{P}(X)$ be an operation on μ and $\mathcal{H}$ be a hereditary class on X. A subset A of X is said to be

(i) $\mu\mathcal{H}g$-closed if $c_\mu(A) \subseteq U$ whenever $A \setminus U^{\gamma_\mu} \in \mathcal{H}$ and $U \in \mu$.

(ii) μg-closed (see [13]) if $c_\mu(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \mu$.

Remark 2.14
If A is $\mu\mathcal{H}g$-closed, then it is μg-closed. Also if $\mathcal{H} = \{\emptyset\}$, then a $\mu\{\emptyset\}g$-closed set is simply a μg-closed set.

Example 2.15
Let $X = \{a, b, c\}$ $\mu = \{\emptyset, X, \{a\}\}$, $\gamma_\mu \colon \mu \to \mathcal{P}(X)$ be an operation defined by $A^{\gamma_\mu} = c_\mu(A)$ for $A \subseteq X$. Let $\mathcal{H} = \{\emptyset, \{b\}\}$. It can be checked that $\{a, b\}$ is a μg-closed set but not a $\mu\mathcal{H}g$-closed set.

Theorem 2.16
Let (X, μ) be a GTS, $\gamma_\mu \colon \mu \to \mathcal{P}(X)$ be an operation on μ and $\mathcal{H}$ be a hereditary class on X. If (X, μ) is $\gamma_\mu\mathcal{H}$-compact and A is a μ-closed set such that $(X \setminus A)^{\gamma_\mu} = X \setminus A$, then A is $\gamma_\mu\mathcal{H}$-compact relative to μ.

Proof. Let A be a μ-closed set and $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of A by μ-open sets of X. Then $(X \setminus A) \cup [\bigcup\{U_\alpha : \alpha \in \Lambda\}]$ is a μ-open cover of X. Thus there exists a finite subset Λ_0 of Λ such that $X \setminus \big[(X \setminus A)^{\gamma_\mu} \cup \big[\bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\}\big]\big] \in \mathcal{H}$. But

$$\begin{aligned}
X \setminus \big[(X \setminus A)^{\gamma_\mu} \cup \big[\textstyle\bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\}\big]\big] & \\
&= [X \setminus (X \setminus A)^{\gamma_\mu}] \cap \big[X \setminus \textstyle\bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\}\big] \\
&= A \cap \big[X \setminus \textstyle\bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\}\big] \\
&= A \setminus \textstyle\bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\}.
\end{aligned}$$

Therefore, $A \setminus \bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\} \in \mathcal{H}$. Thus A is $\gamma_\mu\mathcal{H}$-compact relative to μ.

We call an operation $\gamma_\mu \colon \mu \to \mathcal{P}(X)$ additive (see [16]) if for any $A, B \in \mu$, $(A \cup B)^{\gamma_\mu} = A^{\gamma_\mu} \cup B^{\gamma_\mu}$.

Theorem 2.17
Let (X,μ) be a GTS, $\gamma_\mu\colon \mu \to \mathcal{P}(X)$ be an operation on μ and $\mathcal{H}$ be a hereditary class on X. Let A be a $\mu\mathcal{H}g$-closed subset of X with $A \subseteqq B \subseteqq c_\mu(A)$. If B is $\gamma_\mu\mathcal{H}$-compact relative to μ, then A is $\gamma_\mu\mathcal{H}$-compact relative to μ. If in addition, γ_μ is additive, then converse part is also true.

Proof. Suppose that B is $\gamma_\mu\mathcal{H}$-compact relative to μ and $\{U_\alpha :\ \alpha \in \Lambda\}$ be a cover of A by μ-open sets of X. Since A is a $\mu\mathcal{H}g$-closed subset of X, A is μg-closed and $B \subseteqq c_\mu(A) \subseteqq \bigcup\{U_\alpha :\ \alpha \in \Lambda\}$. Since B is $\gamma_\mu\mathcal{H}$-compact relative to μ, there exists a finite subset Λ_0 of Λ such that $B \setminus \bigcup\{U_\alpha^{\gamma_\mu} :\ \alpha \in \Lambda_0\} \in \mathcal{H}$. Since $A \subseteqq B$, $A \setminus \bigcup\{U_\alpha^{\gamma_\mu} :\ \alpha \in \Lambda_0\} \in \mathcal{H}$. Thus A is $\gamma_\mu\mathcal{H}$-compact relative to μ.

Conversely, suppose that A is $\gamma_\mu\mathcal{H}$-compact relative to μ and $\{U_\alpha :\ \alpha \in \Lambda\}$ be a cover of B by μ-open sets of X. Then $\{U_\alpha :\ \alpha \in \Lambda\}$ is a cover of A by μ-open sets of X. So there exists a finite subset Λ_0 of Λ such that $A \setminus \bigcup\{U_\alpha^{\gamma_\mu} :\ \alpha \in \Lambda_0\} \in \mathcal{H}$. Since A is $\mu\mathcal{H}g$-closed, $c_\mu(A) \subseteqq \bigcup\{U_\alpha :\ \alpha \in \Lambda_0\}$. Since $B \subseteqq c_\mu(A)$, $B \subseteqq \bigcup\{U_\alpha :\ \alpha \in \Lambda_0\} \subseteqq \bigcup\{U_\alpha^{\gamma_\mu} :\ \alpha \in \Lambda_0\}$ and B is γ_μ-compact relative to μ. Hence, B is $\gamma_\mu\mathcal{H}$-compact relative to μ.

3. $\gamma_\mu\mathcal{H}$-compact subsets and related continuity

Throughout the rest of the paper (X,μ) and (Y,λ) will denote GTS's and $\gamma_\mu\colon \mu \to \mathcal{P}(X)$ and $\beta_\lambda\colon \lambda \to \mathcal{P}(Y)$ will denote two operations on μ and λ, respectively.

Definition 3.1
A function $f\colon (X,\mu) \to (Y,\lambda)$ is said to be

(i) (γ,β)-continuous if for each $x \in X$ and each λ-open set V with $f(x) \in V$, there exists a μ-open set U containing x such that $f(U^{\gamma_\mu}) \subseteqq V^{\beta_\lambda}$.

(ii) weakly (γ,β)-continuous if $f^{-1}(V)$ is γ_μ-open for each β_λ-open set V of Y.

Theorem 3.1
Every (γ,β)-continuous function $f\colon (X,\mu) \to (Y,\lambda)$ is weakly (γ,β)-continuous.

Proof. Let V be any β_λ-open subset of Y. For each $x \in f^{-1}(V)$, $f(x) \in V$ and so there exists a λ-open set W containing $f(x)$ such that $f(x) \in W \subseteqq W^{\beta_\lambda} \subseteqq V$. Since f is (γ,β)-continuous, there exists $U \in \mu$ containing x such that $f(U^{\gamma_\mu}) \subseteqq W^{\beta_\lambda}$. Therefore, $x \in U \subseteqq U^{\gamma_\mu} \subseteqq f^{-1}(f(U^{\gamma_\mu})) \subseteqq f^{-1}(W^{\beta_\lambda}) \subseteqq f^{-1}(V)$. Thus f is weakly (γ,β)-continuous.

Example 3.2
Let $X = Y = \{a,b,c\}$, $\mu = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$ and $\lambda = \{\emptyset,Y,\{b\},\{c\},\{b,c\}\}$. Then (X,μ) and (Y,λ) are two GTS's. Let $\gamma_\mu\colon \mu \to \mathcal{P}(X)$ be defined by $A^{\gamma_\mu} = i_\mu(c_\mu(A)))$ for all $A \subseteqq X$. Let $\beta_\lambda\colon \beta \to \mathcal{P}(X)$ be defined by $A^{\beta_\lambda} = c_\lambda(A)$ for any $A \subseteqq X$. Consider the identity function $f\colon (X,\mu) \to (Y,\lambda)$. Then f is a weakly (γ,β)-continuous function but not a (γ,β)-continuous function.

Theorem 3.3
If (Y,λ) is β_λ-regular, then every weakly (γ,β)-continuous function $f\colon (X,\mu) \to (Y,\lambda)$ is (γ,β)-continuous.

Proof. Let $x \in X$ and V be a λ-open subset of Y containing $f(x)$. Then there exists a β_λ-open set W in Y such that $f(x) \in W \subseteqq W^{\beta_\lambda} \subseteqq V$. Since f is weakly (γ, β)-continuous, $f^{-1}(W)$ is a γ_μ-open set containing x and hence a μ-open set in X such that $x \in U \subseteqq U^{\gamma_\mu} \subseteqq f^{-1}(W)$. Therefore, $f(U^{\gamma_\mu}) \subseteqq W \subseteqq W^{\beta_\lambda} \subseteqq V \subseteqq V^{\beta_\lambda}$. Thus f is a (γ, β)-continuous function.

We recall [1] that if $\mathcal{H}$ is a hereditary class on a set X and $f\colon (X, \mu) \to (Y, \lambda)$ is a function, then $f(\mathcal{H}) = \{f(H) : H \in \mathcal{H}\}$ is also a hereditary class on Y.

Theorem 3.4
Let $\mathcal{H}$ be a hereditary class on (X, μ), $f\colon (X, \mu) \to (Y, \lambda)$ be a (γ, β)-continuous function and A be $\gamma_\mu\mathcal{H}$-compact relative to μ. Then $f(A)$ is $\beta_\lambda f(\mathcal{H})$-compact relative to λ.

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be any cover of $f(A)$ by λ-open sets of Y. Then for each $x \in A$, there exists $\alpha(x)$ such that $f(x) \in V_{\alpha(x)}$. Thus there exists a μ-open set $U_{\alpha(x)}$ containing x such that $f(U^{\gamma_\mu}_{\alpha(x)}) \subseteqq V^{\beta_\lambda}$. Now, $\{U_{\alpha(x)} : x \in A\}$ is a cover of A by μ-open sets of X. Thus there exists a finite subset A_0 of A such that $A \setminus \bigcup\{U^{\gamma_\mu}_{\alpha(x)} : x \in A_0\} \subseteqq H$ for some $H \in \mathcal{H}$. Therefore,

$$f(A) \subseteqq \bigcup\{f(U^{\gamma_\mu}_{\alpha(x)}) : x \in A_0\} \cup f(H) \subseteqq \bigcup\{V^{\beta_\lambda}_{\alpha(x)} : x \in A_0\} \cup f(H)$$

Hence,

$$f(A) \setminus \bigcup\{V^{\beta_\lambda}_{\alpha(x)} : x \in A_0\} \in f(\mathcal{H}).$$

Thus $f(A)$ is $\beta_\lambda f(\mathcal{H})$-compact relative to λ.

Theorem 3.5
Let $\mathcal{H}$ be a hereditary class on (X, μ), $f\colon (X, \mu) \to (Y, \lambda)$ be a weakly (γ, β)-continuous function and A be $\gamma_\mu\mathcal{H}$-compact relative to $\gamma_\mu O(X)$. Then $f(A)$ is $\beta_\lambda f(\mathcal{H})$-compact relative to $\beta_\lambda O(Y)$.

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be any cover of $f(A)$ by β_λ-open sets of Y. For each $x \in A$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is weakly (γ, β)-continuous, $x \in f^{-1}(V_{\alpha(x)}) \in \gamma_\mu O(X)$ and $\{f^{-1}(V_{\alpha(x)}) : x \in A\}$ is a cover of A by γ_μ-open sets of X. Since A is $\gamma_\mu\mathcal{H}$-compact relative to $\gamma_\mu O(X)$, there exists a finite subset A_0 of A and $H \in \mathcal{H}$ such that $A \subseteqq \bigcup\{f^{-1}(V_{\alpha(x)}) : x \in A_0\} \cup H$. Hence $f(A) \subseteqq \bigcup\{V_{\alpha(x)} : x \in A_0\} \cup f(H)$. Therefore, $f(A)$ is $\beta_\lambda f(\mathcal{H})$-compact relative to $\beta_\lambda O(Y)$.

Definition 3.2
A function $f\colon (X, \mu) \to (Y, \lambda)$ is said to be (α, β)-closed if for each $y \in Y$ and $U \in \mu$ containing $f^{-1}(y)$, there exists $V \in \lambda$ containing y such that $f^{-1}(V^{\beta_\lambda}) \subseteqq U^{\gamma_\mu}$.

Theorem 3.6
Let $\mathcal{H}$ be a hereditary class on (X, μ) and γ_μ be additive. Let $f\colon (X, \mu) \to (Y, \lambda)$ be a (γ, β)-closed surjection. If $f^{-1}(y)$ is μ-compact relative to μ for each $y \in Y$ and B is $\beta_\lambda\mathcal{H}$-compact relative to λ, then $f^{-1}(B)$ is $\gamma_\mu f^{-1}(\mathcal{H})$-compact relative to μ.

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of $f^{-1}(B)$ by μ-open sets of X. For each $y \in B$, $f^{-1}(y)$ is μ-compact relative to μ, so there exists a finite subset Λ_0 of Λ such that $f^{-1}(y) \subseteqq \bigcup\{U_\alpha : \alpha \in \Lambda_0\} = U_y$ (say). Since U_y is a μ-open set of X containing $f^{-1}(y)$ and f is (γ, β)-closed, there exists a λ-open set V_y containing y such that $f^{-1}(V_y^{\beta_\lambda}) \subseteqq U_y^{\gamma_\mu}$. So $\{V_y : y \in B\}$ is a cover of B by λ-open sets of Y. Since B is $\beta_\lambda\mathcal{H}$-compact relative to λ, there exists a finite subset B_0 of B such that $B \setminus \bigcup\{V_y^{\beta_\lambda} : y \in B_0\} \in \mathcal{H}$. Therefore

$$B \subseteqq \bigcup\{V_y^{\beta_\lambda} : y \in B_0\} \cup H$$

for some $H \in \mathcal{H}$ and hence

$$\begin{aligned} f^{-1}(B) &\subseteqq \bigcup\{f^{-1}(V_y^{\beta_\lambda}) : y \in B_0\} \cup f^{-1}(H) \\ &\subseteqq \bigcup\{U_y^{\gamma_\mu} : y \in B_0\} \cup f^{-1}(H) \\ &\subseteqq \bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0, y \in B_0\} \cup f^{-1}(H). \end{aligned}$$

Thus $f^{-1}(B) \setminus \bigcup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0, y \in B_0\} \in f^{-1}(\mathcal{H})$. Thus $f^{-1}(B)$ is $\gamma_\mu f^{-1}(\mathcal{H})$-compact relative to μ.

References

[1] Carpintero, Carlos, et al. "μ-compactness with respect to a hereditary class." *Bol. Soc. Parana. Mat. (3)* 34, no. 2 (2016): 231-236. Cited on 33, 34, 35 and 40.

[2] Császár, Ákos. "γ-compact spaces." *Acta Math. Hungar.* 87, no. 1-2 (2000): 99-107. Cited on 33.

[3] Császár, Ákos. "Generalized topology, generalized continuity." *Acta Math. Hungar.* 96, no. 4 (2002): 351-357. Cited on 33 and 34.

[4] Császár, Ákos. "Generalized open sets in generalized topologies." *Acta Math. Hungar.* 106, no. 1-2 (2005): 53-66. Cited on 34.

[5] Császár, Ákos. "Modification of generalized topologies via hereditary classes." *Acta Math. Hungar.* 115, no. 1-2 (2007): 29-36. Cited on 33 and 34.

[6] Janković, Dragan S. "On functions with α-closed graphs." *Glasnik Mat.* 18(38) (1983): 141-148. Cited on 33.

[7] Kasahara, Shouro. "Operation-compact spaces." *Math. Japon.* 24, no. 1 (1979/80): 97-105. Cited on 33.

[8] Krishnan, Sai Sundara G., and Maximilian Ganster, and Krishnan Balachandran. "Operation approaches on semi-open sets and applications." *Kochi J. Math.* 2 (2007): 21-33. Cited on 33.

[9] Newcomb, Robert Lewis, Jr *Topologies which are compact modulo an ideal.* Ph.D. Thesis. Santa Barbara: University of California, 1968. Cited on 33.

[10] Ogata, Hayao. "Operations on topological spaces and associated topology." *Math. Japon.* 36, no. 1 (1991): 175-184. Cited on 33.

[11] Qahis, Abdo, and Takashi Noiri. "Functions and weakly $\mu\mathcal{H}$-compact spaces (invited paper)." *Eur. J. Pure Appl. Math.* 10, no. 3 (2017): 410-418. Cited on 33 and 35.

[12] Qahis, Abdo, and Heyam Hussain AlJarrah, and Takashi Noiri. "Weakly μ-compact via a hereditary class." *Bol. Soc. Parana. Mat. (3)* 39, no. 3 (2021): 123-135. Cited on 33.

[13] Roy, Bishwambhar. "On a type of generalized open sets." *Appl. Gen. Top.* 12, no. 2 (2011): 163-173. Cited on 38.

[14] Roy, Bishwambhar, and Saeid Jafari. "On covering properties via generalized open sets." *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* 55 (2012): 57-65. Cited on 35.

[15] Roy, Bishwambhar. "Applications of operations on minimal generalized open sets." *Afr. Mat.* 29, no. 7-8 (2018): 1097-1104. Cited on 33 and 34.

[16] Roy, Bishwambhar, and Takashi Noiri. "Applications on operations on weakly compact generalized topological spaces." *Carpathian Math. Publ.* 12, no. 2 (2020): 461-467. Cited on 35, 36 and 38.

[17] Sarsak, Mohammad S. "Weakly μ-compact spaces." *Demonstratio Math.* 45, no. 4 (2012): 929-938. Cited on 35.

[18] An, Tran Van, and Dang Xuan Cuong, and Haruo Maki. "On operation-preopen sets in topological spaces." *Sci. Math. Jpn.* e-2008, 241-260. Cited on 33.

[19] Zahran, Ahmed M., and Kamal El-Saady, and A. Ghareeb. "Modification of weak structures via hereditary classes." *Appl. Math. Lett.* 25, no. 5 (2012): 869-872. Cited on 33.

Takashi Noiri
2949-1 Shiokita-cho, Hinagu
Yatsushiro-shi, Kumamoto-ken
Japan
E-mail: t.noiri@nifty.com

Bishwambhar Roy
Department of Mathematics
Women's Christian College
6,Greek Church Row
Kolkata-700 026
India
E-mail: bishwambhar_roy@yahoo.co.in

Received: June 28, 2021; final version: June 28, 2021;
available online: September 5, 2022.