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A simpler method to get only the true solutions of cubic and quartic equations using Tschirnhaus transformation

Abstract. The classic method of solving the cubic and the quartic equations using Tschirnhaus transformation yields true as well as false solutions. Recently some papers on this topic are published, in which methods are given to get only the true solutions of cubic and quartic equations. However these methods have some limitations. In this paper the author presents a method of solving cubic and quartic equations using Tschirnhaus transformation, which yields only the true solutions. The proposed method is much simpler than the methods published earlier.

1. Introduction

It is well known that the classic method of solving the cubic and the quartic equations using Tschirnhaus transformation yields true as well as false solutions, and there is no way to identify the true solutions other than the trial and error method [6, 1]. Recently, three papers have appeared on this topic, [3, 4, 5], which present methods for solving the cubic and the quartic equations using Tschirnhaus transformation to get only the true solutions; two papers are on the cubic equations [3, 4], and one paper is on the quartic equations [5].

In [3], a special quadratic Tschirnhaus transformation with only one parameter is defined ($y = (x - c)^2$); however in this approach, all the three true solutions are identified unambiguously only when the discriminant of the cubic equation is positive, else only one true solution is identified, and the remaining two true solutions are to be identified by the trial and error method only.

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The other two papers ([4, 5]) follow a common approach for obtaining only the true solutions from Tschirnhaus method in all situations. The only difference is that the paper [4] deals with the cubic equations, and the paper [5] deals with the quartic equations; so it is sufficient to discuss one paper. Now in the paper [4], again a special quadratic Tschirnhaus transformation is defined ($x^2 = 2cx + d^2 - c^2 + y$); and the true solutions of a cubic equation are obtained in all situations by means of forcing one solution of the transformed cubic equation to be zero, implying $y = 0$, which when substituted in the transformation yields two values of x as: $x = c \pm d$. The method then identifies one of the two values as the true solution after a lengthy and involved algebraic manipulation.

In this paper, the author proposes a much simpler method than the one that is given in the three papers mentioned above, to get only the true solutions of the cubic and quartic equations using Tschirnhaus transformation. The case of a cubic equation is discussed first, followed by that of a quartic equation.

2. Cubic equation

Let us consider the depressed cubic equation,

$$x^3 + ax + b = 0, \tag{1}$$

where the coefficients, a and b, in (1) are complex numbers in general. Let the quadratic Tschirnhaus transformation be,

$$y = x^2 - mx - n, \tag{2}$$

where m and n are the unknown numbers. Expressing (2) as $x^2 = mx + n + y$ and using it in (1) repeatedly yields an expression for x,

$$x = -\frac{m(y+n)+b}{y+m^2+n+a}. \tag{3}$$

Use of (3) in (2) for eliminating x yields a cubic equation in y,

$$\begin{aligned} &y^3 + (3n+2a)y^2 + (3n^2 + 4an + am^2 - 3bm + a^2)y \\ &\qquad + n^3 + 2an^2 + n(am^2 - 3bm + a^2) - bm^3 - abm - b^2 = 0. \end{aligned} \tag{4}$$

Equating the coefficient of y^2 in (4) to zero yields $n = -\frac{2a}{3}$, and consequently (4) becomes,

$$y^3 + \left(am^2 - 3bm - \frac{a^2}{3}\right)y - bm^3 - \frac{2a^2m^2}{3} + abm - b^2 - \frac{2a^3}{27} = 0. \tag{5}$$

Further, equating the coefficient of y in (5) to zero yields a quadratic equation in m,

$$am^2 - 3bm - \frac{a^2}{3} = 0, \tag{6}$$

and solving it yields two values for m,

$$m = \frac{3b}{2a} \pm \sqrt{\frac{27b^2 + 4a^3}{12a^2}}.$$

Due to condition (6), (5) becomes a binomial cubic equation, $y^3 = p^3$, where $p^3 = bm^3 + \frac{2a^2m^2}{3} - abm + b^2 + \frac{2a^3}{27}$. Using (6) p^3 is simplified as:

$$p^3 = \frac{(9bm + 2a^2)(27b^2 + 4a^3)}{27a^2}.$$

The three solutions of $y^3 = p^3$ are,

$$y = p, \quad \frac{p(-1 + \sqrt{3}i)}{2} \quad \text{and} \quad \frac{p(-1 - \sqrt{3}i)}{2}, \tag{7}$$

where p is principal cube-root of p^3 and is given by,

$$p = \left[\frac{(9bm + 2a^2)(27b^2 + 4a^3)}{27a^2}\right]^{1/3}. \tag{8}$$

The three solutions in y, determined in such a way [see (7)], are substituted in (3) to obtain the three expressions in x, which are nothing but the three true solutions of the given cubic equation (1).

Indeed, let x be any solution of (3) and m be any of two numbers satisfying (6). Then there exists $j \in \{0, 1, 2\}$ such that x is given by the formula (3) with y replaced with $y_j = p\omega^j$ where ω is the 3rd root of unity $\left[\omega = \frac{-1 \pm \sqrt{3}i}{2}\right]$ – the last statement does not depend on the choice of m.

2.1. A numerical example on cubic equation

Let us solve one numerical example using the proposed method. Consider the following cubic equation,

$$x^3 - 6x - 9 = 0.$$

Using $n = -\frac{2a}{3}$ we determine $n = 4$. Solving (6) the two values of m are obtained: 4 and 0.5; let $m = 4$. Using these values in (8), p is determined as: -7. With these values of m, n, and p, the three solutions in y are obtained from (7) as: -7, $-3.5(-1 + \sqrt{3}i)$, and $-3.5(-1 - \sqrt{3}i)$. Using these values of y in (3), the three solutions in x are determined as: 3, $-1.5 + 0.5\sqrt{3}i$, and $-1.5 - 0.5\sqrt{3}i$.

Now, using the other value of m (i.e. $m = 0.5$), we determine $p = 3.5$ from (8). So, from (7) the three solutions in y are determined as: 3.5, $1.75(-1 + \sqrt{3}i)$ and $1.75(-1 - \sqrt{3}i)$. Using (3), the three solutions of $x^3 - 6x - 9 = 0$ are obtained as: 3, $-1.5 - 0.5\sqrt{3}i$ and $-1.5 + 0.5\sqrt{3}i$. Notice that all the three solutions determined are true solutions.

3. Quartic equation

Let us now consider the depressed quartic equation,

$$x^4 + ax^2 + bx + c = 0, \tag{9}$$

where a, b, and c are coefficients in (9). Use of quadratic Tschirnhaus transformation (2) in the form of $x^2 = mx + n + y$ in (9) repeatedly yields an expression for x,

$$x = -\frac{y^2 + (m^2 + 2n + a)y + m^2 n + n^2 + an + c}{2my + m^3 + 2mn + am + b}. \tag{10}$$

Use of (10) in (2) for eliminating x yields a quartic equation in y,

$$\begin{aligned} y^4 + (4n + 2a)y^3 + (6n^2 + 6an + am^2 - 3bm + a^2 + 2c)y^2 \\ + [n(4n^2 + 6an + 2am^2 + 4c + 2a^2 - 6bm) \\ + m(4cm - bm^2 - ab) + 2ac - b^2]y \\ + (m^2 n + n^2 + an + c)^2 \\ - (m^3 + 2mn + am + b)(mn^2 + bn - cm) = 0. \end{aligned} \tag{11}$$

Equating the coefficient of y^3 to zero determines n as: $n = -\frac{a}{2}$; and using it in (11) results in,

$$\begin{aligned} y^4 + \Big(am^2 - 3bm - \frac{a^2 - 4c}{2}\Big)y^2 - b\Big(m^3 + \frac{a^2 - 4c}{b}m^2 - 2am + b\Big)y \\ + cm^4 + \frac{ab}{2}m^3 + \frac{a(a^2 - 4c)}{4}m^2 - \frac{b(a^2 - 4c)}{4}m \\ + \Big(\frac{a^2 - 4c}{4}\Big)^2 + \frac{ab^2}{2} = 0. \end{aligned} \tag{12}$$

Further, equating the coefficient of y to zero yields a cubic equation in m, which is known as resolvent cubic equation,

$$m^3 + \frac{a^2 - 4c}{b}m^2 - 2am + b = 0, \tag{13}$$

and also renders (12) as quadratic equation in y^2,

$$y^4 + py^2 + q = 0, \tag{14}$$

where p and q are given by,

$$\begin{aligned} p &= am^2 - 3bm - \frac{a^2 - 4c}{2}, \\ q &= cm^4 + \frac{ab}{2}m^3 + \frac{a(a^2 - 4c)}{4}m^2 - \frac{b(a^2 - 4c)}{4}m + \Big(\frac{a^2 - 4c}{4}\Big)^2 + \frac{ab^2}{2}. \end{aligned} \tag{15}$$

Solving the resolvent cubic equation (13) determines three values of m; any one of these values of m can be used in (12) to get the quadratic equation (14) in y^2, see page 56 in [2] for more details. Solving (14) yields two values of y^2,

$$y^2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}. \tag{16}$$

Taking square root of (16) yields four values of y,

$$y = \pm\sqrt{\frac{-p \pm \sqrt{p^2 - 4q}}{2}}. \tag{17}$$

Now, the four values of y obtained from (17) are substituted in (10) to obtain the four values of x, which are the true solutions of the given quartic equation (9).

3.1. A numerical example on quartic equation

Let us solve the quartic equation,

$$x^4 - 2x^2 + 4x - 3 = 0,$$

using the proposed method. The relation $n = -\frac{a}{2}$ determines n as $n = 1$. The resolvent cubic equation (13) is: $m^3 + 4m^2 + 4m + 4 = 0$, which, when solved, yields three values of m as: $-3.13039543\ldots$, $-0.43480228\ldots + 1.04342743\ldots i$ and $-0.43480228\ldots - 1.04342743\ldots i$.

Let $m = -3.13039543\ldots$; using (15) we determine p and q as: $9.96599406\ldots$ and $-193.6882805\ldots$. From (17), the four solutions in y are determined as: $3.13039543\ldots$, $-3.13039543\ldots$, $4.44582609\ldots i$, and $-4.44582609\ldots i$. Using these values of y in (10), the four solutions of the quartic equation, $x^4 - 2x^2 + 4x - 3 = 0$, are obtained as: 1, $-2.13039543\ldots$, $0.56519771\ldots + 1.04342743\ldots i$, and $0.56519771\ldots - 1.04342743\ldots i$, which are the four true solutions.

The other two values of m also yield the same solutions in x as obtained above, see [2]. The author encourages the reader to verify that this is in fact true.

4. Discussion

The methods followed in the earlier papers mentioned in Section 1 ([3, 4, 5]) invariably consist of obtaining two values of x for the each solution of the transformed equation, and then attempting to identify the true solutions, which resulted in a lengthy and involved procedure. But in the proposed method presented here, only one value of x is determined by the repeated elimination of higher powers of x from the given cubic (or quartic as the case may be) equation and the transformation. This has resulted in one-to-one correspondence between y and x values eliminating the ambiguity in x values, which in turn has yielded only the true solutions in a simpler manner.

5. Conclusions

A much simpler method is presented in this paper than the earlier methods for solving the cubic and quartic equations using Tschirnhaus transformation, which yields only the true solutions. The limitations of the earlier methods are discussed.

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