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Solution to algebraic equations of degree 4 and the fundamental theorem of algebra by Carl Friedrich Gauss

Abstract. Since *Geronimo Cardano*, algebraic equations of degree 4 have been solved analytically. Frequently, the solution algorithm is given in its entirety. We discovered two algorithms that lead to the same *resolvente*, each with two solutions; therefore, six formal solutions appear to solve an algebraic equation of degree four. Given that a square was utilized to derive the solution in both instances, it is critical to verify each solution. This check reveals that the four Cardanic solutions are the only four solutions to an algebraic equation of degree four. This demonstrates that *Carl Friedrich Gauss*' (1799) *fundamental theorem of algebra* is not simple, despite the fact that it is a fundamental theorem. This seems to be a novel insight.

1. Introduction and preliminaries

Although algebra has been regarded the kings' discipline of mathematics for over four thousand years, not every year a contribution to nonlinear algebra is made.

For this article we need the following definitions and theorems.

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1.1. Polynomial sum

A *polynomial sum* of an algebraic equation of degree n is given by the following equation of variable x and constant coefficients A_μ,

$$\sum_{\mu=0}^{n} A_\mu x^\mu = 0, \qquad A_n \neq 0. \tag{1}$$

1.2. Polynomial product

The *product representation* of an algebraic equation of degree n is considered to be with constant coefficients b_μ,

$$A_n \prod_{\mu=1}^{n} (x - b_\mu) = 0, \qquad A_n \neq 0. \tag{2}$$

Remark 1.1
Equation (2) changes to (1) by expanding the product. The way from (1) to (2) is done by solving an algebraic equation. Until now a general solution using just power terms and basic calculation has been found to (1) for $n \in \{1, 2, 3, 4\}$ only.

1.3. Normal form

The *normal form* of an algebraic equation is built by division of A_n for both (1) and (2) with $a_\mu = \frac{A_\mu}{A_n}$,

$$x^n + \sum_{\mu=0}^{n-1} a_\mu x^\mu = 0,$$
$$\prod_{\mu=1}^{n} (x - b_\mu) = 0.$$

Remark 1.2
In a *normal form*, the degree n of an algebraic equation is independent of the concrete values of all parameters, like $\{A_\mu, a_\mu, b_\mu\}$.

1.4. Reduced sum

The *reduced sum* of an algebraic equation reduces the normal sum by the substitution $x \to y - \frac{a_{n-1}}{n}$ via the *binomial theorem*

$$y^n + y^{n-1} a_{n-1} \left(1 - \frac{n}{n}\right) + \sum_{\mu=0}^{n-2} c_\mu y^\mu = y^n + \sum_{\mu=0}^{n-2} c_\mu y^\mu. \tag{3}$$

1.5. Reduced product

With w_μ being dependent on the expected roots z_μ of the corresponding *resolvente equation* of degree $n-1$, the following approach of a *reduced product form* [5, equation (10), page 4],

$$\prod_{\nu=0}^{n-1}\Big(y-\sum_{\mu=1}^{n-1} w_\mu \mathrm{e}^{\frac{2\mathrm{i}\pi\mu\nu}{n}}\Big) \tag{4}$$

leads by multiplication to a *reduced algebraic sum* of the type (3).
To show this, the sum coefficient c_{n-1} is calculated as follows, when y^{n-1} is multiplied by the sum of all terms being independent of y,

$$\begin{aligned} c_{n-1} &= \sum_{\nu=0}^{n-1}\sum_{\mu=1}^{n-1}(-w_\mu)\mathrm{e}^{\frac{2\mathrm{i}\pi\mu\nu}{n}} = -\sum_{\mu=1}^{n-1} w_\mu \sum_{\nu=0}^{n-1}\big(\mathrm{e}^{\frac{2\mathrm{i}\pi\mu}{n}}\big)^\nu \\ &= -\sum_{\mu=1}^{n-1} w_\mu \frac{\mathrm{e}^{2\mathrm{i}\pi\mu}-1}{\mathrm{e}^{\frac{2\mathrm{i}\pi\mu}{n}}-1} = -\sum_{\mu=1}^{n-1} w_\mu \frac{1-1}{\mathrm{e}^{\frac{2\mathrm{i}\pi\mu}{n}}-1} = 0. \end{aligned} \tag{5}$$

Remark 1.3
The result (5) is valid for integer $n>1$. The closed polygone path in the complex number area with result $c_{n-1}=0$ is typical for each algebraic problem [2, keyword *algêbro*, page 71]. The fact that a product representation leads to a reduced polynomial sum does not prove that the approach (4) gives the correct structure of the solution, which will be demonstrated during the rest of this article for the example $n=4$.

2. Cubic solutions

Due to Johann Faulhaber (1604) [4, chapter 1.1.2, page 22] the solution to a *reduced equation* of degree 3,

$$y^3+py+q=0$$

is given by [1, section 2.4.2.3., page 132] with $\mu\in\{1,2,3\}$,

$$y_\mu = 2\frac{\sqrt{-p}}{\sqrt{3}}\cos\Big(\frac{1}{3}\Big(2\pi\mu-\arccos\Big(\frac{3q}{2p}\frac{\sqrt{3}}{\sqrt{-p}}\Big)\Big)\Big). \tag{6}$$

Remark 2.1
The representation (6) is valid for any complex value $p\neq 0$ and any complex q, too. For $p=0$, a *cubic root* results with $\mu\in\{1,2,3\}$,

$$y_\mu = \sqrt[3]{-q}\,\mathrm{e}^{\frac{2\mathrm{i}\mu}{3}}.$$

The representation (6) is better to deal with, than the historical Cardanic solution (9) representation [1, section 2.4.2.3., page 131], which goes back to Niccolo Tartaglia [3, section V., page 109].

Since $\cos(x)$ and $\arccos(y)$ both are *quadratic* functions, the result (6) is mainly just another representation of the historical Cardanic solution (9), which can be shown by the following equivalent equations

$$
\begin{aligned}
y = \cos(x) &= \frac{e^{ix} + e^{-ix}}{2}, && (7)\\
2ye^{ix} &= (e^{ix} + e^{-ix})e^{ix},\\
(e^{ix})^2 - 2y(e^{ix}) + 1 &= 0,\\
\exp(ix) = y \pm \sqrt{y^2-1} &= y \pm i\sqrt{1-y^2},\\
x = \arccos(y) &= \frac{\ln(y \pm i\sqrt{1-y^2})}{i}. && (8)
\end{aligned}
$$

The terms (8) and (7), applied to the result (6), yield with $X = \frac{3\sqrt{3}q}{2p\sqrt{-p}}$,

$$
\begin{aligned}
y_3 &= 2\frac{\sqrt{-p}}{\sqrt{3}}\,\frac{e^{\frac{\ln(X\pm\sqrt{X^2-1})}{3}} + e^{-\frac{\ln(X\pm\sqrt{X^2-1})}{3}}}{2}\\
&= \frac{\sqrt{-p}}{\sqrt{3}}\left(\sqrt[3]{X \pm \sqrt{X^2-1}} + \frac{1}{\sqrt[3]{X \pm \sqrt{X^2-1}}}\right)\\
&= \frac{\sqrt{-p}}{\sqrt{3}}\left(\sqrt[3]{X \pm \sqrt{X^2-1}} + \frac{\sqrt[3]{X \mp \sqrt{X^2-1}}}{\sqrt[3]{X^2-(X^2-1)}}\right) && (9)\\
&= \frac{\sqrt{-p}}{\sqrt{3}}\left(\sqrt[3]{\frac{3\sqrt{3}q}{2p\sqrt{-p}} \pm \sqrt{\left(\frac{27q^2}{-4p^3}\right)-1}} + \sqrt[3]{\frac{3\sqrt{3}q}{2p\sqrt{-p}} \mp \sqrt{\left(\frac{27q^2}{-4p^3}\right)-1}}\right)\\
&= \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} \mp \sqrt{\frac{q^2}{4}+\frac{p^3}{27}}},
\end{aligned}
$$

which is the Cardanic solution [3, section V., page 109] of Niccolo Tartaglia.

3. Ambiguity for degree 4

Expanding a product of n algebraic solutions in x always leads to an algebraic sum in x of degree n. The *fundamental theorem of algebra* by Carl Friedrich Gauss is concerning the inverse operation of this. Therefore, especially the beginners of nonlinear algebra may ask, whether the *fundamental theorem of algebra* would be a trivial theorem or not. Where are the examples of ambiguity?

A first answer to this question seems to be possible for $n = 4$, for which Cardano and his crew have found analytical solutions. Here, the ambiguity is given by the possibilities to build a reduced product, for example as follows

$$\begin{aligned}&(y - w_1 - w_2 - w_3)\\&(y - w_1 + w_2 + w_3)\\&(y + w_1 - w_2 + w_3)\\&(y + w_1 + w_2 - w_3) = 0.\end{aligned} \tag{10}$$

The coefficient c_3 of y^3 in the resulting algebraic sum (3) of degree 4 is just the sum of all terms being independent of the solution y and dependent to the root containing functions $\{w_1, w_2, w_3\}$. Here, this sum turns out to be zero, because the signs $+$ and $-$ occur each twice for all three so far unknown root containing functions $\{w_1, w_2, w_3\}$,

$$\begin{aligned}c_3 = &\ \ (-w_1 - w_2 - w_3)\\&+(-w_1 + w_2 + w_3)\\&+(+w_1 - w_2 + w_3)\\&+\underline{(+w_1 + w_2 - w_3)}\\= &\ \ (+\ 0\ +\ 0\ +\ 0) = 0.\end{aligned}$$

The approach (10) is different from the approach (4), thus there is a need to look carefully for the details. The main difference is, that in (10) there are no signs beside $+$ and $-$, whereas in (4) there are signs $+\mathrm{i} = +\sqrt{-1}$ and $-\mathrm{i} = -\sqrt{-1}$, too.

A root *function* usually maps the complex number area to a part of the complex number area, whereas a root *relation* maps the complex number area to the whole complex number area. The properties of the so far known solutions to algebraic equations show by calculating corresponding checks, that all Cardanic solutions work with unambiguos root *functions*, whereas the power notation by Leibniz may be used further on when discussing ambiguous root *relations*. This problem shall be discussed in another publication some day, as well as a complete mapping of all possible solution paths to algebraic equations of degree 4.

Here, the use of the approaches (10) and (4) would lead to an algebraic equation of degree 6 in w, where an interpretation as a normalized cubic equation in $z = 4w^2$ is possible. Therefore, the Cardanic (10) and the alternative (4) approaches are both discussed with explicit root *functions* in the following chapters to get unambiguity.

Carl Friedrich Gauss may have found similar results, but the German literature of this time omits illustrating examples. Another idea is, that Gauss would not have wanted to correct Leibniz notation within his doctoral thesis, because such an elaboration is a beginner's work. Now, an *example* to the fundamentality of his theorem is needed.

4. Algebraic products of degree 4

A *reduced product* is given by the product of the four historical Cardanic solutions (10) [1, section 2.4.2.3., page 133] with $w_\mu = -\frac{\sqrt{z_\mu}}{2}$ and independent of the approach (4),

$$\begin{aligned}&\Big(y-\frac{+\sqrt{z_1}+\sqrt{z_2}-\sqrt{z_3}}{2}\Big)\Big(y-\frac{+\sqrt{z_1}-\sqrt{z_2}+\sqrt{z_3}}{2}\Big)\\&\Big(y-\frac{-\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3}}{2}\Big)\Big(y-\frac{-\sqrt{z_1}-\sqrt{z_2}-\sqrt{z_3}}{2}\Big)=0.\end{aligned}\tag{11}$$

With the same substitution $w_\mu \to -\frac{\sqrt{z_\mu}}{2}$, the approach (4) yields

$$\begin{aligned}&\Big(y-\frac{-\sqrt{z_1}-\sqrt{z_2}-\sqrt{z_3}}{2}\Big)\Big(y-\frac{-\mathrm{i}\sqrt{z_1}+\sqrt{z_2}+\mathrm{i}\sqrt{z_3}}{2}\Big)\\&\Big(y-\frac{+\sqrt{z_1}-\sqrt{z_2}+\sqrt{z_3}}{2}\Big)\Big(y-\frac{+\mathrm{i}\sqrt{z_1}+\sqrt{z_2}-\mathrm{i}\sqrt{z_3}}{2}\Big)=0.\end{aligned}\tag{12}$$

The two results (11) and (12) have got two common factors

$$\begin{aligned}&\Big(y-\frac{+\sqrt{z_1}-\sqrt{z_2}+\sqrt{z_3}}{2}\Big)\Big(y-\frac{-\sqrt{z_1}-\sqrt{z_2}-\sqrt{z_3}}{2}\Big)\\&\quad=y^2+y\sqrt{z_2}+\Big(\frac{-z_1+z_2-z_3-2\sqrt{z_1z_3}}{4}\Big)=0.\end{aligned}\tag{13}$$

The two rest factors of the *reduced Cardanic product* (11) are

$$\begin{aligned}&\Big(y-\frac{+\sqrt{z_1}+\sqrt{z_2}-\sqrt{z_3}}{2}\Big)\Big(y-\frac{-\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3}}{2}\Big)\\&\quad=y^2-y\sqrt{z_2}+\Big(\frac{-z_1+z_2-z_3+2\sqrt{z_1z_3}}{4}\Big)=0.\end{aligned}\tag{14}$$

The two rest factors of the *alternative reduced product* (12) are

$$\begin{aligned}&\Big(y-\frac{-\mathrm{i}\sqrt{z_1}+\sqrt{z_2}+\mathrm{i}\sqrt{z_3}}{2}\Big)\Big(y-\frac{+\mathrm{i}\sqrt{z_1}+\sqrt{z_2}-\mathrm{i}\sqrt{z_3}}{2}\Big)\\&\quad=y^2-y\sqrt{z_2}+\Big(\frac{z_1+z_2+z_3-\sqrt{z_1z_3}}{4}\Big)=0.\end{aligned}\tag{15}$$

5. Algebraic Sums of Degree 4

The product of all four Cardanic factors (11) is given by the product of (13) and (14),

$$\begin{aligned}&y^4-y^2\Big(\frac{z_1+z_2+z_3}{2}\Big)+y(\sqrt{z_1z_2z_3})+\Big(\frac{(-z_1+z_2-z_3)^2}{16}-\frac{z_1z_3}{4}\Big)=0\\&\Leftrightarrow\ y^4-y^2\Big(\frac{z_1+z_2+z_3}{2}\Big)+y(\sqrt{z_1z_2z_3})\\&\qquad+\Big(\frac{z_1^2+z_2^2+z_3^2}{16}-\frac{z_1z_2+z_1z_3+z_2z_3}{8}\Big)=0.\end{aligned}\tag{16}$$

Thus the Cardanic approach (11) yields a reduced polynomial sum of degree 4.

The product of all four alternative factors (12) is given by the product of (13) and (15),

$$y^4 - y^2\Big(\frac{z_2}{2} + \sqrt{z_1 z_3}\Big) + y\sqrt{z_2}\Big(\frac{z_1 + z_3}{2}\Big) + \Big(\frac{(z_2 - 2\sqrt{z_1 z_3})^2 - (z_1 + z_3)^2}{16}\Big) = 0, \quad (17)$$

which is equivalent to

$$y^4 - y^2\Big(\frac{z_2}{2} + \sqrt{z_1 z_3}\Big) + y\sqrt{z_2}\Big(\frac{z_1 + z_3}{2}\Big) + \Big(\frac{z_2^2 - 4z_2\sqrt{z_1 z_3} - (z_1 - z_3)^2}{16}\Big) = 0. \quad (18)$$

Thus the alternative approach (12) yields a reduced polynomial sum of degree 4. The *reduced sum* of an algebraic equation of degree 4 is given by

$$y^4 + py^2 + qy + r = 0. \quad (19)$$

Remark 5.1
By comparison of the coefficients follow several determining equation sets.

6. Determining equation sets

By comparison of the coefficients of the equations (16) and (19) result the Cardanic determining equations of degree 4,

$$\begin{cases} r = \dfrac{z_1^2 + z_2^2 + z_3^2}{16} - \dfrac{z_1 z_2 + z_1 z_3 + z_2 z_3}{8} \\ q = \sqrt{z_1 z_2 z_3} \\ p = -\dfrac{z_1 + z_2 + z_3}{2}. \end{cases} \quad (20)$$

Eliminating two of the three unknown roots z_1, z_2, and z_3 leads to the *resolvente* polynomial of degree 3.
By comparison of the coefficients of the equations (18) and (19) result the alternative determining equations of degree 4,

$$\begin{cases} r = \dfrac{z_2^2}{16} - z_2\dfrac{\sqrt{z_1 z_3}}{4} - \dfrac{(z_1 - z_3)^2}{16} \\ q = \dfrac{\sqrt{z_2}}{2}(z_1 + z_3) \\ p = -\dfrac{z_2}{2} - \sqrt{z_1 z_3}. \end{cases} \quad (21)$$

Within the *Cardanic determining equations* (20), any variable z_1, z_2, and z_3 leads to the same structure of the solving path. Therefore, all three equations are solved to the variable z_3,

$$\begin{cases} z_3 = z_1 + z_2 \pm 2\sqrt{z_1 z_2 + 4r} \\ z_3 = \dfrac{q^2}{z_1 z_2} \\ z_3 = -2p - z_1 - z_2. \end{cases} \quad (22)$$

Now, the last solution of (22) is set into the other equations of (22),

$$\begin{cases} z_2^2 + z_2(2p + z_1) + (z_1 + p)^2 - 4r = 0 \\ z_2^2 + z_2(2p + z_1) + \dfrac{q^2}{z_1} = 0. \end{cases} \tag{23}$$

The difference of the two equations (23) is independent of the variable z_2 and leads to the *cubic resolvente equation* [1, section 2.4.2.3., page 133]

$$(z_1 + p)^2 - 4r - \frac{q^2}{z_1} = 0 \quad \Leftrightarrow \quad z_1^3 + 2pz_1^2 + (p^2 - 4r)z_1 - q^2 = 0. \tag{24}$$

Within the *alternative determining equations* (21), the variable z_2 leads to the most simple system of solutions. Therefore, all three equations are solved to the variable z_2,

$$\begin{cases} z_2 = 2\sqrt{z_1 z_3} \pm \sqrt{(z_1 + z_3)^2 + 16r} \\ z_2 = \dfrac{4q^2}{(z_1 + z_3)^2} \\ z_2 = -2p - 2\sqrt{z_1 z_3}. \end{cases} \tag{25}$$

Here, the expressions of z_1 and z_3 in the first solution of (25) or in (17) and (19) for $y \to 0$ are found in the last two equations of (25),

$$\begin{cases} (z_2 - 2\sqrt{z_1 z_3})^2 - (z_1 + z_3)^2 - 16r = 0 \\ (z_1 + z_3)^2 = \dfrac{4q^2}{z_2} \\ -2\sqrt{z_1 z_3} = 2p + z_2. \end{cases}$$

By this the *resolvente* equation can be constructed directly [1, section 2.4.2.3., page 133],

$$(2z_2 + 2p)^2 - \frac{4q^2}{z_2} - 16r = 0 \quad \Leftrightarrow \quad z_2^3 + 2pz_2^2 + (p^2 - 4r)z_2 - q^2 = 0. \tag{26}$$

The result is, that both resolvente equations (24) and (26) are identical in z [1, section 2.4.2.3., page 133],

$$z^3 + 2pz^2 + (p^2 - 4r)z - q^2 = 0.$$

Therefore, now six different solutions (13), (14), and (15) have been found to be solution candidates for a reduced algebraic equation (19).

Remark 6.1
However, since a square was needed during each solving path, the calculation of a *check* for each of these solutions is necessary!

7. Calculating the check

The resolvente equation (6) is an algebraic equation of degree 3 and leads by the substitution $z \to \zeta - \frac{2p}{3}$ to a reduced algebraic equation

$$\zeta^3 - \Big(\frac{p^2}{3} + 4r\Big)\zeta + \frac{8rp}{3} - \frac{2p^3}{27} - q^2 = 0.$$

Thus the solution (6) yields

$$\zeta_\mu = \sqrt{\frac{4}{3}\Big(\frac{p^2}{3} + 4r\Big)} \cos\left(\frac{2\pi\mu}{3} - \frac{1}{3}\arccos\left(\frac{3\sqrt{3}\big(\frac{8rp}{3} - \frac{2p^3}{27} - q^2\big)}{-2\big(\frac{p^2}{3} + 4r\big)\sqrt{\frac{p^2}{3} + 4r}}\right)\right). \tag{27}$$

The check of this solution (27) can be done conveniently via some substitutions

$$\begin{aligned} z_\mu &= a + b\cos\Big(\frac{2\pi\mu}{3} - \frac{1}{3}\arccos\Big(-\frac{4c}{b^3}\Big)\Big) \\ &= a + b\cos\Big(\frac{2\pi\mu}{3}\Big)\cos(d) - b\sin\Big(\frac{2\pi\mu}{3}\Big)\sin(d) \\ &= a + b\cos(e). \end{aligned} \tag{28}$$

Applying (28) to (6) yields by $\cos^3(e) = \frac{3\cos(e)}{4} + \frac{\cos(3e)}{4}$ [1, section 2.5.2.1.5., page 183] the following equivalent equations

$$\begin{aligned} &a^3 + 3a^2b\cos(e) + 3ab^2\cos^2(e) + b^3\cos^3(e) \\ &\quad + 2p(a^2 + 2ab\cos(e) + b^2\cos^2(e)) + (p^2 - 4r)(a + b\cos(e)) - q^2 = 0, \end{aligned}$$

$$\begin{aligned} &b^3\Big(\frac{3\cos(e)}{4} + \frac{\cos(3e)}{4}\Big) + \cos^2(e)(3ab^2 + 2pb^2) \\ &\quad + \cos(e)(3a^2b + 4abp + bp^2 - 4br) + a^3 + 2a^2p + ap^2 - 4ar - q^2 = 0, \end{aligned}$$

$$b\cos(e)\Big(\frac{3b^2}{4} + \frac{4p^2}{3} - \frac{8p^2}{3} + p^2 - 4r\Big) - c - \frac{8p^3}{27} + \frac{8p^3}{9} - \frac{2p^3}{3} + \frac{8rp}{3} - q^2 = 0,$$

which is equivalent to

$$\begin{cases} a = -\dfrac{2p}{3} \\ b = \sqrt{\dfrac{4}{3}\Big(\dfrac{p^2}{3} + 4r\Big)} \\ c = \dfrac{8rp}{3} - \dfrac{2p^3}{27} - q^2. \end{cases} \tag{29}$$

Equation (6) becomes correct for the values (29) of a, b, and c, which are the substitutions in (28).

Remark 7.1
Since the equation sets (20) and (21) contain three unknown roots of a polynomial of degree 3, it is possible to solve these equations *directly* by the approach (28) without constructing the resolvente equation (6) before. This method allows to find the parameters a, b, and c successively instead of at once. There is hope, that some day this method of using already known properties of algebraic solutions will help to solve even more complicated algebraic equations.

The three solutions (28) to the *resolvente equation* (6) are for $\mu \in \{1, 2, 3\}$,

$$\begin{cases} z_1 = a - \frac{b}{2}\cos(d) - \frac{\sqrt{3}b}{2}\sin(d) \\ z_2 = a - \frac{b}{2}\cos(d) + \frac{\sqrt{3}b}{2}\sin(d) \\ z_3 = a + b\cos(d), \end{cases}$$

thus

$$z_1 + z_2 + z_3 = 3a = -2p. \tag{30}$$

The squares of the roots z_μ are

$$\begin{aligned} z_{1,2}^2 &= \left(a - \frac{b}{2}\cos(d) \mp \frac{\sqrt{3}b}{2}\sin(d)\right)^2 \\ &= a^2 + \frac{b^2}{4}\cos^2(d) + \frac{3b^2}{4}(1 - \cos^2(d)) - ab\cos(d) \\ &\quad \mp \sqrt{3}ab\sin(d) \pm \frac{\sqrt{3}b^2}{2}\sin(d)\cos(d) \\ &= a^2 + \frac{3b^2}{4} - \frac{b^2}{2}\cos^2(d) - ab\cos(d) + \mp\sqrt{3}ab\sin(d) \\ &\quad \pm \frac{\sqrt{3}b^2}{2}\sin(d)\cos(d), \\ z_3^2 &= a^2 + b^2\cos^2(d) + 2ab\cos(d), \end{aligned}$$

thus

$$z_1^2 + z_2^2 + z_3^2 = 3a^2 + \frac{3b^2}{2} = 2(p^2 + 4r). \tag{31}$$

The mixed products are

$$\begin{aligned} z_1 z_2 &= \left(a - \frac{b}{2}\cos(d)\right)^2 - \frac{3b^2}{4}(1 - \cos^2(d)) \\ &= a^2 - \frac{3b^2}{4} + b^2\cos^2(d) - ab\cos(d), \end{aligned} \tag{32}$$

$$\begin{aligned} z_{1,2} z_3 &= \left(a - \frac{b}{2}\cos(d) \mp \frac{\sqrt{3}b}{2}\sin(d)\right)(a + b\cos(d)) \\ &= a^2 + \frac{ab}{2}\cos(d) \mp \frac{\sqrt{3}ab}{2} - \frac{b^2}{2}\cos^2(d) \mp \frac{\sqrt{3}b^2}{2}\sin(d)\cos(d). \end{aligned} \tag{33}$$

Thus,

$$z_1 z_2 + z_1 z_3 + z_2 z_3 = 3a^2 - \frac{3b^2}{4} = p^2 - 4r. \tag{34}$$

Finally, a threefold product is calculated via (32),

$$\begin{aligned} z_1 z_2 z_3 &= \Big(a^2 - \frac{3b^2}{4} + b^2\cos^2(d) - ab\cos(d)\Big)(a + b\cos(d)) \\ &= a^3 - \frac{3ab^2}{4} + ab^2\cos^2(d) - a^2 b\cos(d) + a^2 b\cos(d) \\ &\quad - \frac{3b^3}{4}\cos(d) + \frac{b^3}{4}\Big(3\cos(d) - \frac{4c}{b^3}\Big) - ab^2\cos^2(d) \\ &= a^3 - \frac{3ab^2}{4} - c = q^2. \end{aligned} \tag{35}$$

Now the approach (28) within the Cardanic equation set (20) yields via the results (30), (31), (34) and (7),

$$\begin{cases} 16r = 3a^2 + \dfrac{3b^2}{2} - 2\Big(3a^2 - \dfrac{3b^2}{4}\Big) \\ q = \sqrt{a^3 - \dfrac{3ab^2}{4} - c} \\ -2p = 3a. \end{cases}$$

This result is *without* any trigonometric expressions, thus here the approach (28) has turned out to be a *key* to simplify the Cardanic equation set (20), the solutions of which are

$$\begin{cases} a = -\dfrac{2p}{3} \\ b = +\sqrt{\dfrac{3a^2 + 16r}{3}} = +\sqrt{\dfrac{4}{3}\Big(\dfrac{p^2}{3} + 4r\Big)} \\ c = a^3 - \dfrac{3ab^2}{4} - q^2 = \dfrac{8rp}{3} - \dfrac{2p^3}{27} - q^2. \end{cases} \tag{36}$$

The sign before the second equations of (36) is $+$, because in the trigonometric view of (27) this is an absolute value. The parameters a, b, and c are the same as in (29).

Now the approach (28) within the alternative equation set (21) via the results (32) and (33) yields the insight, that an expression containing trigonometric expressions will remain under the square root $\sqrt{z_1 z_3}$, thus the task is very complicated

$$\begin{cases} r = \dfrac{(z_2 - 2\sqrt{z_1 z_3})^2}{16} - \dfrac{(z_1 + z_3)^2}{16} \\ q = \dfrac{\sqrt{z_2}}{2}(z_1 + z_3) \\ p = -\dfrac{z_2}{2} - \sqrt{z_1 z_3}. \end{cases}$$

There are two cases to discuss an expression $\sqrt{z_1 z_3}$ in general, because the sequential order of the roots z_1, z_2, and z_3 should be arbitrary

$$\sqrt{z_1 z_2} = \sqrt{a^2 - \frac{3b^2}{4} + b^2\cos^2(d) - ab\cos(d)},$$

$$z_{1,2}z_3 = a^2 + \frac{ab}{2}\cos(d) \mp \frac{\sqrt{3}ab}{2}\sin(d) - \frac{b^2}{2}\cos^2(d) \mp \frac{\sqrt{3}b^2}{2}\sin(d)\cos(d).$$

The problem can occur that a solution does not fit to the corresponding set of determining equations, since there was a square during the solution path. Now the remaining of trigonometric expressions leads to the suggestion, that the factors (15) will not stand the check in the reduced equation (19).

Remark 7.2
As a consequence, an approach like (28) can check, whether the non–algebraic expressions within an algebraic determination equation set are removed, or not.

8. Repeating the check

The common factors (13) yield the following check

$$y = \frac{\pm\sqrt{z_1} - \sqrt{z_2} \pm \sqrt{z_3}}{2},$$

$$\begin{aligned} y^2 &= \frac{z_1 + z_2 + z_3}{4} + \frac{\mp\sqrt{z_1 z_2} + \sqrt{z_1 z_3} \mp \sqrt{z_2 z_3}}{2} \\ &= -\frac{p}{2} + \frac{\mp\sqrt{z_1 z_2} + \sqrt{z_1 z_3} \mp \sqrt{z_2 z_3}}{2}, \end{aligned}$$

$$\begin{aligned} y^4 &= \frac{p^2}{4} + \frac{z_1 z_2 + z_1 z_3 + z_2 z_3}{4} - \frac{p\left(\mp\sqrt{z_1 z_2} + \sqrt{z_1 z_3} \mp \sqrt{z_2 z_3}\right)}{2} \\ &\quad + \frac{\sqrt{z_1 z_2 z_3}\left(\mp\sqrt{z_1} + \sqrt{z_2} \mp \sqrt{z_3}\right)}{2} \\ &= \frac{p^2}{4} + \frac{p^2 - 4r}{4} - \frac{p\left(\mp\sqrt{z_1 z_2} + \sqrt{z_1 z_3} \mp \sqrt{z_2 z_3}\right)}{2} - qy \\ &= -py^2 - qy - r. \end{aligned}$$

Therefore the check in (19) is fulfilled.
The Cardanic rest factors (14) yield the following check

$$y = \frac{\pm\sqrt{z_1} + \sqrt{z_2} \mp \sqrt{z_3}}{2}.$$

$$\begin{aligned} y^2 &= \frac{z_1 + z_2 + z_3}{4} + \frac{\pm\sqrt{z_1 z_2} - \sqrt{z_1 z_3} \mp \sqrt{z_2 z_3}}{2} \\ &= -\frac{p}{2} + \frac{\pm\sqrt{z_1 z_2} - \sqrt{z_1 z_3} \mp \sqrt{z_2 z_3}}{2}. \end{aligned}$$

$$\begin{aligned} y^4 &= \frac{p^2}{4} + \frac{z_1z_2 + z_1z_3 + z_2z_3}{4} - \frac{p(\pm\sqrt{z_1z_2} - \sqrt{z_1z_3} \mp \sqrt{z_2z_3})}{2} \\ &\quad + \frac{\sqrt{z_1z_2z_3}(\mp\sqrt{z_1} - \sqrt{z_2} \pm \sqrt{z_3})}{2} \\ &= \frac{p^2}{4} + \frac{p^2 - 4r}{4} - \frac{p(\pm\sqrt{z_1z_2} - \sqrt{z_1z_3} \mp \sqrt{z_2z_3})}{2} - qy \\ &= -py^2 - qy - r. \end{aligned}$$

Therefore the check in (19) is fulfilled.
The alternative rest factors (15) yield the following check

$$\begin{aligned} y &= \frac{\mp\mathrm{i}\sqrt{z_1} + \sqrt{z_2} \pm \mathrm{i}\sqrt{z_3}}{2}, \\ y^2 &= \frac{-z_1 + z_2 - z_3}{4} + \frac{\mp\mathrm{i}\sqrt{z_1z_2} + \sqrt{z_1z_3} \pm \mathrm{i}\sqrt{z_2z_3}}{2}. \end{aligned} \tag{37}$$

Now there are two possibilities to calculate the constant expression in (37),

- $\{z_1, z_2\}$ or $\{z_2, z_3\}$ is a complex conjugated pair.
- $\{z_1, z_3\}$ is a complex conjugated pair.

The first case yields

$$f(a, b, d) = -z_1 + z_2 - z_3 = -a - b\cos(d) \pm \sqrt{3}b\sin(d).$$

The second case yields

$$f(a, b, d) = -z_1 + z_2 - z_3 = -a + 2b\cos(d).$$

Both cases can be calculated as follows

$$\begin{aligned} y^4 &= \frac{f(a, b, d)^2}{16} + \frac{f(a, b, d)(\mp\mathrm{i}\sqrt{z_1z_2} + \sqrt{z_1z_3} \pm \mathrm{i}\sqrt{z_2z_3})}{4} \\ &\quad + \frac{-z_1z_2 + z_1z_3 - z_2z_3}{4} + \frac{\sqrt{z_1z_2z_3}(\mp\mathrm{i}\sqrt{z_1} + \sqrt{z_2} \pm \mathrm{i}\sqrt{z_3})}{2} \\ &= g(a, b, d)y^2 + qy + h(a, b, d). \end{aligned}$$

Thus there is no possibility to fulfill (19) for $y \neq 0$ and $q \neq 0$.

Remark 8.1
Therefore the alternative rest factors (15) lead into a fallacy.

9. Conclusion

An algebraic equation of degree n has got exactly n solutions.

This *fundamental theorem of algebra* by Carl Friedrich Gauss is a non–trivial theorem, because already for $n = 4$ there is a possibility of at least 6 formal solutions leading to the same resolvente equation (6), but only 4 of these solutions stand the check. There is a need of good examples to demonstrate a theorem. Now this task has been done for the *fundamental theorem of algebra*, which has been the aim of this article.

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