Ann. Univ. Paedagog. Crac. Stud. Math. 21 (2022), 71-92
DOI: 10.2478/aupcsm-2022-0007

FOLIA 355

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica 21 (2022)

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On a certain characterisation of the semigroup of positive natural numbers with multiplication

Abstract. In this paper we continue our investigation concerning the concept of a *liken*. This notion has been defined as a sequence of non-negative real numbers, tending to infinity and closed with respect to addition in $\mathbb{R}$. The most important examples of likens are clearly the set of natural numbers $\mathbb{N}$ with addition and the set of positive natural numbers $\mathbb{N}^*$ with multiplication, represented by the sequence $(\ln(n+1))_{n=0}^{\infty}$. The set of all likens can be parameterized by the points of some infinite dimensional, complete metric space. In this *space of likens* we consider elements up to isomorphism and define *properties of likens* as such that are isomorphism invariant. The main result of this paper is a theorem characterizing the liken $\mathbb{N}^*$ of natural numbers with multiplication in the space of all likens.

1. Introduction

We begin by recalling the content of the paper [4], which is necessary to formulate and prove the main result of this paper, i.e. Theorem 25. As it was mentioned in [4], the notion of a liken may be considered as some way of talking about the so called *Beurling numbers* [1]. The family $\mathcal{H}$ of all likens (we say also *the space of likens*) described in [4], constitutes a kind of a *natural environment* where "live" the two fundamental mathematical structures: $(\mathbb{N}, +)$, the natural numbers with addition, and $(\mathbb{N}^*, \cdot)$, the natural numbers with multiplication, which as mathematical structures are ordered semigroups (monoids). Let us note here, that according to the definition, by a liken we mean a sub-semigroup of the additive semigroup $\mathbb{R}^+$.

AMS (2010) Subject Classification: 11A41.
Keywords and phrases: Beurling numbers, distribution of prime numbers, Cauchy translation equation, numerical semigroups, Apéry sets.
ISSN: 2081-545X, e-ISSN: 2300-133X.

For this reason we replace $\mathbb{N}^*$ by the sequence $(\ln(n+1))_{n=0}^{\infty}$, however without changing the notation, and we call elements of this last sequence also the natural numbers. Although the set of likens is rather big, being infinite dimensional complete metric space, the most of likens seem to be of little interest and if they were brought to life in [4], it was only to look at the liken $\mathbb{N}^*$ from a slightly different point of view.

One may ask at this point whether it is worth building such a sophisticated structure as the space of likens mentioned above, if only two elements of this space are worth attention. It seems so, because these two noteworthy elements, i.e. the likens $\mathbb{N}$ and $\mathbb{N}^*$ may be considered as two pillars on which all mathematics is based and hence there is never too much knowledge about them. The value of the approach presented here is clearly limited by the fact that we assume the set of real numbers as known, but nevertheless it seems that the characterizations of the likens $\mathbb{N}$ and $\mathbb{N}^*$ presented in this paper which will be formulated and proved below, are not entirely trivial.

The exact definition of a liken (in different versions) will be recalled in the next section, but at the beginning it is enough to know, that a liken $\mathbb{L}$ is a strictly increasing sequence $\mathbb{L} = (x_n)_{n=0}^{\infty}$ of real numbers, which is a sub-semigroup of the semigroup $\mathbb{R}^+$. Hence in each liken $\mathbb{L}$ we have two types of mathematical structures inherited from $\mathbb{R}^+$, i.e. the algebraic structure of the sub-semigroup with addition and the structure of the ordered space with respect to the natural order in $\mathbb{R}$. This make it possible to define the isomorphism of likens as a bijection which preserves both, algebraic and ordinal structures.

Different details concerning the relation of the isomorphism of likens will be discussed in the next section. It turns out, and this is in a sense a typical situation, that all interesting likens (infinitely generated and with uniqueness) are algebraically isomorphic to each other, and at the same time, they are always isomorphic to each other as ordered spaces. On the other hand, as it was proved in [4], they are isomorphic as likens if and only if their sets of generators are homothetic. As it was mentioned above, this situation is typical. To better understand what we mean by the term "typical", let us consider the example of the family of all infinite dimensional, separable Banach spaces. Each two such spaces are isomorphic as vector spaces since they have the vector bases of the same cardinality, and each two such spaces are homeomorphic as topological spaces by the theorem of Kadec-Anderson [2]. However, two such spaces are isomorphic as Banach spaces only when there exists a linear isomorphism which is also a topological homeomorphism. It seems that the basic advantage of using the abstract language of likens lies in the fact that we can formulate different properties of likens in this language and consequently distinguish between them. Roughly speaking, *a property of a liken* is such property, which is preserved by isomorphisms of likens. Perhaps the most important of such properties of likens is that they are generated by their indecomposable elements (just like natural numbers by prime numbers in the semigroup $\mathbb{N}^*$), but this property is common for all likens. We will provide non-trivial examples of a few such properties later in the paper, but for now let us consider the following (trivial) example. Let $\mathbb{L} = (x_{n=0}^{\infty})$ be a liken and let us consider the property: *the element x_2 is indecomposable.* This property is a property of likens which is fulfilled in $\mathbb{N}^*$ since $x_2 = 3$ and 3 is a prime number, but is not true in $\mathbb{N}$, where $x_2 = 2 = 1 + 1$ and in $\mathbb{N}$ only $x_1 = 1$ is indecomposable.

The main result of this paper is Theorem 25 which gives a characterization of the liken $\mathbb{N}^*$ among all likens. For this, first we will formulate two properties of likens called *concavity*, denoted by (C), and *Ockham's razor property* denoted by (OR). Then the main theorem states: *if a liken* $\mathbb{L}$ *has the properties* (C) *and* (OR) *then it is isomorphic to* $\mathbb{N}^*$*.*

The paper is organized as follows. In Section 2 we recall some definitions, notations and theorems proved in [4], which will be used in this paper. In fact the content of Section 2 is to be found in [4], but because of some small differences in notations and since the paper [4] is quite long, we collect in Section 2 all we will need to know about likens in this paper. In Section 3 we formulate a number of general properties of likens, in particular the mentioned properties (C) and (OR). All these properties are reformulations in the language of likens of the known properties of natural numbers. In Section 4 we present the proof of Theorem 25. The last section contains a number of remarks.

2. Definitions, notations and the main results about likens

In this paper, as in [4], we will use the following notations:

$$\begin{aligned}
\mathbb{Q}^+ &= [0,\infty) \cap \mathbb{Q}, \\
\mathbb{R}^{\mathbb{N}} &= \{\overrightarrow{a} = (a_i)_1^\infty :\ a_i \in \mathbb{R}\}. \\
(\mathbb{R}^+)^{\mathbb{N}} &= \{\overrightarrow{a} \in \mathbb{R}^{\mathbb{N}} :\ a_i \geq 0\}. \\
\mathbb{R}_0^{\mathbb{N}} &= \{\overrightarrow{a} \in \mathbb{R}^{\mathbb{N}} :\ \exists j : i > j \Rightarrow a_i = 0\}. \\
\mathbb{Q}^{\mathbb{N}} &= \{\overrightarrow{a} = (a_i)_1^\infty :\ a_i \in \mathbb{Q}\}. \\
\mathbb{N}_0^{\mathbb{N}} &= \{\overrightarrow{m} \in \mathbb{N}^{\mathbb{N}} :\ \exists j : i > j \Rightarrow m_i = 0t\}.
\end{aligned} \tag{1}$$

Moreover, for $\overrightarrow{a} \in \mathbb{R}^{\mathbb{N}}$, and for $\overrightarrow{m} \in \mathbb{N}_0^{\mathbb{N}}$ we set

$$\langle \overrightarrow{a}, \overrightarrow{m} \rangle = m_1 a_1 + m_2 a_2 + \cdots .$$

Let us note, that although $\overrightarrow{a}$ may tend to infinity, the right-hand side sum is always finite, since the sequence $\overrightarrow{m}$ in fact is finite.

The definition of a liken given in [4] is the following

Definition 1
A liken $\mathbb{L}$ is a sequence $(x_n)_{n=0}^\infty$ of real numbers such that:

a) $x_0 = 0$ and for all $n \in \mathbb{N}$ we have $x_n < x_{n+1}$,

b) for all $n, m \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that $x_n + x_m = x_k$.

As it was observed in [4], a liken $\mathbb{L}$ is an increasing sequence of nonnegative real numbers, which is closed with respect to the addition and tends to infinity. We have mentioned in Introduction, that likens are sub-semigroups of the semigroup $\mathbb{R}^+$. Since we put $x_0 = 0$ each liken is a monoid.

Now we recall the notion of the isomorphism of likens.

Definition 2
Let $(\mathbb{G}, +)$ be a monoid and let $\mathbb{L}$ be a liken. We will say that a map $\varphi\colon \mathbb{G} \to \mathbb{L}$ is

a) an algebraic homomorphism, when $\varphi(x + y) = \varphi(x) + \varphi(y)$,

b) an algebraic monomorphism, when it is an injective homomorphism,

c) an algebraic isomorphism, when it is a surjective monomorphism.

Thus in particular we know now, what it means that two likens $\mathbb{L}$ and $\mathbb{K}$ are algebraically isomorphic. It is also clear, that each two likens are isomorphic as ordered spaces, since they are similar to the (well) ordered space $(\mathbb{N}, \leq)$. Let us mention, that the map $\mathbb{K} \ni x_n \to y_n \in \mathbb{L}$ is not (in general) a homomorphism of likens, and let us mention also, that if $\varphi\colon \mathbb{K} \to \mathbb{L}$ is an ordinal isomorphism, then it is unique. Finally we set

Definition 3
Two likens $\mathbb{L}$ and $\mathbb{K}$ are isomorphic if the (unique) ordinal isomorphism is also an algebraic homomorphism.

A very important consequence of the axioms of liken is the existence of *indecomposable elements* (called also *irreducible elements* or *prime elements*).

Definition 4
Let $\mathbb{L}$ be a liken and let $u \in \mathbb{L}$. We will say that $u > 0$ is indecomposable if

$$(u = v + w, v \in \mathbb{L}, w \in \mathbb{L}) \Longrightarrow v = 0 \vee w = 0.$$

The following was observed in [4],

Proposition 5
Each liken $\mathbb{L} = (x_n)_0^\infty$ has at least one indecomposable element.

Also (see [4]),

Proposition 6
Let $\mathbb{L}$ be a liken, and let $\mathcal{P}_{\mathbb{L}}$ be the set of all indecomposable elements of $\mathbb{L}$. Then each element of $x \in \mathbb{L}$ can be written in the form

$$x = m_1 \cdot a_1 + m_2 \cdot a_2 + \cdots + m_k \cdot a_k,$$

where $m_1, m_2, \dots, m_k \in \mathbb{N}$, $a_1, a_2, \dots, a_k \in \mathcal{P}_{\mathbb{L}}$, and $k \in \mathbb{N}$.

One may ask now about the uniqueness of the representation from Proposition 6. In general, as it was discussed in [4], such representations are not unique. So the above Definition 1 of a liken admits likens without uniqueness. This is for example the case of the so-called *numerical semigroups* [3] with the associated *Apéry sets* (see also Remark 32 in Section 5). However, in this paper we will be interested mostly only in likens with uniqueness, so further, in this paper, "liken" means usually "liken with uniqueness". This implies, and it will be discussed later, that all infinite dimensional likens with uniqueness are isomorphic algebraically. We recall below shortly the notions and the properties we need below. The detailed description is presented widely in [4].

Let $\mathbb{N}_0^{\mathbb{N}}$, as in (1), denote the set of all sequences of natural numbers, with almost all terms vanishing, i.e.

$$\mathbb{N}_0^{\mathbb{N}} := \{\overrightarrow{n} = (n_1, n_2, \dots) : \ (n_j \in \mathbb{N}) \wedge (\exists i \in \mathbb{N} : k > i \Longrightarrow n_k = 0)\}.$$

In the set $\mathbb{N}_0^{\mathbb{N}}$ we may consider the operations: "+" – addition and "$\cdot$" – multiplication by natural numbers, defined as usually in a cartesian product. With these operations $\mathbb{N}_0^{\mathbb{N}}$ is an algebraic structure, which may be called *semimodule* or *a cone* over $\mathbb{N}$.

We set $e_k = (0, 0, \dots, 0, 1, 0, \dots)$, i.e. e_k is an element of $\mathbb{N}_0^{\mathbb{N}}$, with all terms equal 0 except the $k - th$, which is 1. So we have for $\overrightarrow{n} \in \mathbb{N}_0^{\mathbb{N}}$,

$$\overrightarrow{n} = (n_1, n_2, \dots) = n_1 \cdot e_1 + n_2 \cdot e_2 + \cdots .$$

Using the terminology from the linear algebra we may say, that $(e_k)_1^\infty$ *is a basis of the cone* $\mathbb{N}_0^{\mathbb{N}}$. This means precisely, that each element from $\mathbb{N}_0^{\mathbb{N}}$ can be, in a unique way, written as a linear combination of $(e_k)_{k \in \mathbb{N}}$ with the coefficient from $\mathbb{N}$. Clearly $\mathbb{N}_0^{\mathbb{N}}$ with addition is not only a semigroup, but it is a monoid.

It is obvious that $\mathbb{R}^+$ is a cone over $\mathbb{N}$. A map $\varphi \colon \mathbb{N} \to \mathbb{R}^+$ is a homomorphism of monoids (or of cones), when

$$\varphi(n_1 \cdot e_1 + n_2 \cdot e_2 + \cdots) = n_1 \cdot \varphi(e_1) + n_2 \cdot \varphi(e_2) + \cdots .$$

It is evident, that a homomorphism $\varphi \colon \mathbb{N}_0^{\mathbb{N}} \to \mathbb{R}^+$ cannot be an epimorphism, since $\mathbb{N}_0^{\mathbb{N}}$ is countable, but $\mathbb{R}^+$ is uncountable. However, there exist monomorphisms $\varphi \colon \mathbb{N}_0^{\mathbb{N}} \to \mathbb{R}^+$, and the mentioned above *space of likens* can be considered as space of all such monomorphisms. We will now give a description of this situation (for details see [4]). It follows from the above and from the equality

$$\langle \overrightarrow{a}, \overrightarrow{n} + \overrightarrow{m} \rangle = \langle \overrightarrow{a}, \overrightarrow{n} \rangle + \langle \overrightarrow{a}, \overrightarrow{m} \rangle,$$

that

Proposition 7
Each function $a \colon \mathbb{N} \to \mathbb{R}^+$ can be extended in a unique way to a homomorphism $\tilde{a} \colon \mathbb{N}_0^{\mathbb{N}} \to \mathbb{R}^+$ by linearity, i.e. $\tilde{a}(\overrightarrow{n}) = \langle \overrightarrow{a}, \overrightarrow{n} \rangle$.

Now we will recall the description of the space of infinitely generated likens which was outlined in [4]. It must be clearly stated here that this description will be not used in the proof of the main result of this paper and is only intended to illustrate how large the family of likens in question is. Let $\overrightarrow{a} = (a_i)_1^\infty$ be a sequence of positive real numbers. The liken generated by this sequence will be denoted by $\mathbb{L}_a$. Let us make the following observation

Proposition 8
The liken $\mathbb{L}_a$ is a liken with uniqueness if and only if the sequence $\overrightarrow{a} = (a_i)_1^\infty$ is linearly independent in the vector space $(\mathbb{R}, \mathbb{Q})$.

Proof. Suppose that $\overrightarrow{a}$ is linearly independent and suppose that two linear combinations with natural coefficients are equal

$$\langle \overrightarrow{a}, \overrightarrow{n} \rangle = \langle \overrightarrow{a}, \overrightarrow{m} \rangle.$$

Hence

$$\langle \overrightarrow{a}, \overrightarrow{n} - \overrightarrow{m} \rangle = 0.$$

But $\overrightarrow{n} - \overrightarrow{m}$ is in $\mathbb{Q}^{\mathbb{N}}$ thus $\overrightarrow{n} - \overrightarrow{m} = 0$, i.e $\overrightarrow{n} = \overrightarrow{m}$.

Conversely, suppose that $\mathbb{L}_a$ is a liken with uniqueness and suppose that some linear combination of $\overrightarrow{a}$ with coefficients from $\mathbb{Q}^{\mathbb{N}}$ equals 0, i.e. we have $(p_i, q_i \in \mathbb{Z}, q_i > 0)$,

$$\frac{p_1}{q_1} a_1 + \cdots + \frac{p_k}{q_k} a_k = 0.$$

Let $M = q_1 q_2 \ldots q_k$ and let $m_i = \frac{M}{q_i}$. It follows from the above, that

$$\sum_i m_i p_i a_i = 0,$$

or equivalently

$$\sum_{p_i > 0} m_i p_i a_i = \sum_{p_j < 0} -m_j p_j a_j.$$

The uniqueness implies then that for each i we have $m_i p_i a_i = 0$ hence $p_1 = p_2 = \ldots = p_k = 0$ and this ends the proof.

In other words, we have the following

Proposition 9
Let $\overrightarrow{a} = (a_i)_1^\infty$ be a sequence from $(\mathbb{R}^+)^{\mathbb{N}}$, which is linearly independent in the vector space $(\mathbb{R}, \mathbb{Q})$ and tends to infinity. Then $\tilde{a}(\mathbb{N}_0^{\mathbb{N}})$ is a liken with uniqueness.

Let us recall also one more theorem from [4]. Suppose, that we have two sequences $\overrightarrow{a} = (a_k)_1^\infty$ and $\overrightarrow{b} = (b_k)_1^\infty$, which generate two likens with uniqueness denoted by $\mathbb{L}_a$ and $\mathbb{L}_b$ respectively. We have the following

Theorem 10
In the notations as above the likens $\mathbb{L}_a$ and $\mathbb{L}_b$ are isomorphic, if and only if there exists a positive number λ such that $\overrightarrow{a} = \lambda \cdot \overrightarrow{b}$.

We will analyze now the correspondence $\overrightarrow{a} \to \mathbb{L}_a$ in more details, to establish in what sense this correspondence is one to one. As we have observed in Proposition 6, given a set of generators (finite or infinite) $\overrightarrow{a} = (a_k)_1^\infty$, the liken $\mathbb{L}_a$ does not depend on the sequence $(a_k)_1^\infty$ but depends only on the set of its elements. The only property we need from $(a_k)_1^\infty$ is to be locally finite. Clearly each finite set is locally finite, and for infinite sequences $\overrightarrow{a} = (a_k)_1^\infty$ of generators, it is evident, that such a sequence is locally finite if and only if $\lim_{k\to+\infty} a_k = +\infty$. In other words, for a liken $\mathbb{L}_a$ we can always assume (and we do it in particular in this paper) that its sequence of generators is strictly increasing. Moreover, it follows from Theorem 10 that if we consider likens only up to isomorphism, we may assume without loss of generality, that $a_1 = 1$. We can summarize the above considerations as follows. Let $\mathcal{H}$ denote the space of all infinitely generated likens with uniqueness and let $\mathcal{R}$ denote the space of all sequences $\overrightarrow{a} = (a_i)_1^\infty$ of positive real numbers, which are strictly increasing, linearly independent in $(\mathbb{R}, \mathbb{Q})$, tends to infinity and $a_1 = 1$. Then the correspondence

$$\mathcal{R} \ni \overrightarrow{a} \to \mathbb{L}_a \in \mathcal{H}$$

is a bijection. It seems that this observation ends the description of the space $\mathcal{H}$ from a strictly set-theoretic point of view. Some additional information can be obtained analyzing the space of likens from a topological point of view. As it was observed in [4], the space $\mathcal{R}$ is a subspace of the metric space $(\mathbb{R}^{\mathbb{N}}, d)$ equipped with the standard metric of coordinate-wise convergence. The closure $\overline{\mathcal{R}}$ in $(\mathbb{R}^{\mathbb{N}}, d)$ is then a complete metric space. A more detailed analysis shows that $\mathcal{R}$ is a G_δ set in $\overline{\mathcal{R}}$. In consequence the space of all likens $\mathcal{H}$ is, with respect to the topology of coordinate-wise convergence, a Polish space.

3. Some properties of likens

Let $\mathbb{L} = (x_n)_{n=0}^\infty$ be a liken. By a *property of likens* we will mean, roughly speaking, all conditions concerning likens and formulated using only the language (and properties) of the addition and order in $\mathbb{R}$ and the addition and order in $\mathbb{N}$. Clearly, the properties of likens are preserved by isomorphisms of likens. We present below a few examples of such properties. The method of a construction of these properties is as follows. We consider a particular liken (for example $\mathbb{N}^*$), and a particular property of this liken (for example the twin primes conjecture), we formulate this property in the language of likens, and this way we obtain a property of a liken.

Property 11
We will say that the dimension of $\mathbb{L}$ equals $k \in \mathbb{N}$ if $\mathbb{L}$ has exactly k indecomposable elements. In other words, $\dim(\mathbb{L}) = k \iff \operatorname{card}(\mathcal{P}_\mathbb{L}) = k$. Such a liken will be called *finitely generated*.

Let us mention here, that $\mathbb{N}$ is a one dimensional liken and this is a unique liken with this property (up to isomorphism). In other words, if two likens are finitely generated and are isomorphic, then they have the same dimension, but the converse is not necessarily true, see [4] for details. The so-called *numerical semigroups* (for definition see [3]) are finite dimensional likens. A numerical semigroup is a semigroup generated in $(\mathbb{N}, +)$ by the complements of finite sets. For example the set $\mathbb{N} \setminus \{1, 2\}$ is a numerical semigroup which is a three dimensional liken and its generators are $\{3, 4, 5\}$. This liken is a liken without uniqueness, since, for example $8 = 3 + 5 = 4 + 4$. This way we have

Property 12
Suppose that $\overrightarrow{a} = (a_k)_1^\infty$ is a set of generators (finite or infinite) of the liken $\mathbb{L}$. As we have mentioned above, the liken $\mathbb{L}$ has *the uniqueness property* if for each $x \in \mathbb{L}$ there exists exactly one $\overrightarrow{n} \in \mathbb{N}_0^{\mathbb{N}}$ such that $x = \langle \overrightarrow{a}, \overrightarrow{n} \rangle$.

Property 13
Suppose, that $\mathbb{P} = \{1 = p_1 < p_2 < \ldots\}$ is a subset of the set of natural numbers (finite or infinite). We will say that $\mathbb{L}$ *has its generators exactly in* $\mathbb{P}$ when for each $n \in \mathbb{N}$ we have: $x_n \in \mathbb{L}$ is indecomposable if and only if $n \in \mathbb{P}$.

It is not hard to see, that if $\mathbb{P}$ is finite then one always can find a liken which has the generators exactly in $\mathbb{P}$. When the set $\mathbb{P}$ is infinite then the problem of the existence of a liken which has its generators exactly in $\mathbb{P}$ is more complicated. There is an obvious necessary condition for such a property, namely the set $\mathbb{N} \setminus \mathbb{P}$ must be infinite.

Remark 14
Writing this paper the author did not know if the infinitness condition is also sufficient. Then, one of the anonymous referees proposed the proof of sufficiency of this condition. Below we present this proof, written in details and with the notation changed to match this of the paper.

To show that infinitness condition is sufficient, we have to prove the following theorem.

Theorem 15
If the set $\mathbb{P} \subset \mathbb{N}$ *is such that the set* $\mathbb{N} \setminus \mathbb{P}$ *is infinite then there exists a liken* $\mathbb{L}$ *which has its generators exactly in* $\mathbb{P}$*.*

Proof. We will start by constructing a sequence $(x_n)_0^\infty$, which will turn out to be the wanted liken $\mathbb{L}$. The construction of $(x_n)_0^\infty$ runs inductively. Before describing this construction, we recall a definition used in the formulation of Ockham razor property. Namely, if $\mathbb{L}(x_1, x_2, \ldots, x_n)$ is a liken generated by a sequence $x_1 < x_2 < \ldots < x_n$ then

$$z(x_n) = \min\{x \in \mathbb{L}(x_1, x_2, \ldots, x_n) : \ x > x_n\}.$$

We set $x_0 = 0$ and $x_1 = 1$. Assuming that $x_1, x_2, \ldots, x_n$ have already been constructed we define x_{n+1} as follows: if $n+1 \in \mathbb{N} \setminus \mathbb{P}$ then we set $x_{n+1} = z(x_n)$ and if $n+1 \in \mathbb{P}$ then as x_{n+1} we take any number in the interval $(x_n, z(x_n))$ that is linearly independent with $x_1, x_2, \ldots, x_n$ over $\mathbb{Q}$. We see that this sequence is strictly increasing, but we do not know if $\mathbb{L}$ is a liken. For this we must prove that $\mathbb{L}$ is closed under addition and that it tends to infinity. Both these properties will be proved together inductively. We will say that a number X is composed if $X = x_i + x_j$ for some $1 \le i \le j < n$.

For $N = 1, 2 \ldots$, let T_N be as follows.

T_N : There exists $k_N \in \mathbb{N}$ such that:

$$(W_N) : x_{k_N} = N$$

and if a number X is composed and $X \leq N$ then

$$(V_N) : X \in \mathbb{L}.$$

Let us notice that if all theorems T_N are true, then $\mathbb{L}$ is closed under addition and $\mathbb{L}$ tends to infinity. Clearly T_1 is trivially true, so we start by proving T_2. Observe that $z(x_1) = 2$ and that there are no composed numbers in $(1, 2)$. Thus if an element $x_j \in \mathbb{L}$ is such that $1 < x_j < 2$ then x_j is indecomposable and all indecomposable elements x_j in $(1, 2)$ form a sequence of consecutive indecomposable elements. But each such sequence must be finite. Let x_{k-1} be the biggest indecomposable element in $(1, 2)$. Then $z(x_{k-1}) = z(x_1)$ since each sum of the form $x_i + x_j$ where $1 < i, j < k - 1$ are bigger then 2 . Hence $x_k = 2$. Thus (W_2) and (V_2) are true. Further we will need the following observation.

Observation 16
Let $x_1, x_2, \ldots, x_r \in \mathbb{L}$. Suppose that u and v are two consecutive elements of the liken $\mathbb{L}(x_1, x_2, \ldots, x_r)$ such that $x_r = u$ and such that $x_r = u < v \leq u + 1$ (hence v is composed). Then $v \in \mathbb{L}$.

Indeed, since $u = x_r$ and v are two consecutive elements of $\mathbb{L}(x_1, x_2, \ldots, x_r)$ then we have $z(x_r) = v$. Hence if $r + 1 \in \mathbb{N} \setminus \mathbb{P}$ then $v = x_{r+1}$ and the Observation is true. In consequence it is sufficient to consider only the situation when between x_r and v are only indecomposable elements. But in such a case these indecomposable elements form a sequence of consecutive elements of $\mathbb{L}$ which, by the assumption on $\mathbb{P}$, is finite. Suppose then that $x_{r+1}, x_{r+2}, \ldots, x_{r+s}$ are all indecomposable elements in the interval (x_r, v). This implies (because of $|x_r - v| < 1$) that $x_{r+1}, x_{r+2}, \ldots, x_{r+s}$ do not participate in the computation of $z(x_{r+s})$, or in other words, $z(x_{r+s}) = z(x_r) = v$. In consequence, $v = x_{r+s+1}$ and this ends the proof of Observation.

Now we will prove the induction step. We assume, that T_N is true, and we want to prove T_{N+1}. We know that there exists $m = k_N \in \mathbb{N}$ is such that $x_m = N$ and all sums $x_i + x_j$ are in $\mathbb{L}$ provided that $i, j < m$ and $x_i + x_j \leq N$. Suppose that $v_1 < v_2 < \cdots < v_j$ are all elements of the liken $\mathbb{L}(x_1, x_2, \ldots, x_m)$ which are in $(N, N+1)$. Since they are all bigger than x_m then they are all composed. Now we apply Observation for $u = x_m$ and $v = v_1$ and we obtain that there is m_1 such that $v_1 = x_{m_1}$. Next we use Observation for $u = x_{m_1}$ and $v = v_2$ and we obtain m_2 such that $v_2 = x_{m_2}$ and finally we find m_j such that $v_j = x_{m_j}$. To end the proof we apply Observation for $u = x_{m_j}$ and $v = N + 1$ and we find k_{N+1} such that $x_{k_{N+1}} = N + 1$. Let us observe also, that for each sum of the form $x_p + x_q$ such that $x_p < N$, $x_q < N$ and $x_p + x_q < N + 1$ there exists $i \leq j$ such that $x_p + x_q = v_i = x_{m_i}$. Thus T_{N+1} is true, and Theorem 15 is proved.

Remark 17
In the presented proof we choose the consecutive generators in order to obtain each time a sequence linearly independent over $\mathbb{Q}$. This assures that the constructed liken is a liken with uniqueness. If we drop the claiming of the independence over $\mathbb{Q}$ the proof remains valid, but the constructed liken may not be a liken with uniqueness.

Let us recall here, that Definition 1 does not assure the uniqueness, and that in this paper, as it was mentioned above, for simplicity, we mean *liken* as *liken with uniqueness*. If $\mathbb{L}$ is a liken with uniqueness then each element $x \in \mathbb{L}$ of this liken can be identified with a sequence of its coefficients in the representation

$$x = \langle \overrightarrow{a}, \overrightarrow{n} \rangle = n_1 a_1 + n_2 a_2 + \cdots = n_1(x) a_1 + n_2(x) a_2 + \cdots .$$

We set $\mathrm{supp}(x) = \{i \in \mathbb{N} : \ n_i(x) \neq 0\}$ and we call this set *the support of* x.

Property 18
We will say that a liken $\mathbb{L}$ has *a disjoint support property* if for each $n \in \mathbb{N}$ we have $\mathrm{supp}(x_n) \cap \mathrm{supp}(x_{n+1}) = \emptyset$.

Clearly, if a liken $\mathbb{L}$ has the disjoint support property, then it is infinitely dimensional. Indeed, if a liken $\mathbb{L}$ is generated be the generators $a_1, a_2, \ldots, a_k$ and $x_m = a_1 + a_2 + \cdots + a_k$ then the supports of x_m and x_{m+1} are not disjoint. On the other hand, for example the liken $\mathbb{N}^*$ has this property. If $i \in \mathrm{supp}(x)$ then we will say that a_i divides x (denoted $a_i | x$).

Property 19
We will say that $\mathbb{L}$ has *the parity property* if for each $n \in \mathbb{N}$ we have $(x_1 | x_n) \Leftrightarrow \neg(x_1 | x_{n+1})$.

In [4] we studied the sequence of *gaps in likens*, i.e. the sequence of differences $\delta_{\mathbb{L}}(k) = \delta_k = \delta(x_k) = x_{k+1} - x_k$. By the definition of a liken the sequence δ_k is strictly positive and as it was observed in [4], if $\dim(\mathbb{L}) \geq 2$ then $\lim_{k\to\infty} \delta_{\mathbb{L}}(k) = 0$. However, in general (e.g. in the case of finite dimensional likens) the sequence δ_k is not strictly decreasing. On the other hand there are the likens, for example $\mathbb{N}^*$ such that $\delta_{\mathbb{L}}(k)$ is strictly decreasing. Since the property δ_k *is strictly decreasing* is equivalent to: *for each* $k \in \mathbb{N}$ *we have* $\delta_k > \delta_{k+1}$ or equivalently, $2x_{k+1} > x_k + x_{k+2}$, then we formulate the property of concavity as follows.

Property (C)
A liken $\mathbb{L}$ is said to be *concave* if and only if for each $k \in \mathbb{N}$ the following inequality holds

$$2x_{k+1} > x_k + x_{k+2}.$$

It is not hard to observe, that the liken $\mathbb{L}$ is concave if and only if the points (k, x_k) lie on the graph of a concave function $f\colon [0, \infty) \to \mathbb{R}$. For example $\mathbb{N}^*$ is a concave liken, since the function $x \to \ln(x)$ is concave. Given a sequence $(x_n)_{n=0}^{\infty}$, the sequence of its gaps plays a role of the derivative of the given sequence, thus the condition that the sequence of gaps is strictly decreasing corresponds for differentiable functions to the claiming, that their second derivative is negative.

It is clear, that the concavity of a liken is preserved by isomorphisms, but one may give an example of two concave likens $\mathbb{K}$ and $\mathbb{L}$ which are not isomorphic. Indeed, let $\mathbb{L} = \mathbb{N}^* = (x_n)_{n=0}^{\infty}$ and let $\mathbb{N}^{**} = (y_n)_{n=0}^{\infty} = \mathbb{K} = (\ln(2n+1))_{n=0}^{\infty}$ be a liken of all odd natural numbers (the product of two odd numbers is odd). These likens are both concave, but they are not isomorphic. Indeed, x_3 is composed and y_3 is indecomposable. The same is true if one considers the likens $\mathbb{K}_p = (\ln(pn+1))_{n=0}^{\infty}$ for $p = 1, 2, \ldots$.

Let us look at the liken $\mathbb{N}^*$ and notice that except for one case, of two consecutive elements of this liken, at most one is indecomposable. The situation is different in the liken $\mathbb{N}^{**}$, where every indecomposable twin primes (in $\mathbb{N}^*$) are consecutive elements of the liken $\mathbb{N}^{**}$. We can therefore formulate the following property of likens.

Property 20
We will say that in liken $\mathbb{L}$ *almost all indecomposable elements are separated* when the number of such pairs (x_n, x_{n+1}) in which both elements are indecomposable, is finite.

We may ask if there exist likens without Property 20. The liken $\mathbb{N}^{**}$ is a very natural, however conditional example, because depending on the twin prime conjecture. But in the large space of likens one may find an (unconditional) example of a liken with an infinite sequence of pairs of non-separated generators. We present below a sketch of the proof of the existence of such a liken. We start by choosing an infinitely generated liken $\mathbb{L}_1 = (x_{0,1}, x_{1,1}, \ldots)$ with uniqueness, where $x_{1,1} = 1$ and we choose an arbitrary constant M, for example let us fix $M = 10$. Since $\mathbb{L}_1$ is infinitely generated then there exists a number k_1 such that $x_{k_1,1}$ is a generator of $\mathbb{L}_1$ and $x_{k_1,1} > M_1 = 10$. Now we choose a number $A_1 \in (x_{k_1,1}, x_{k_1+1,1})$ which is linearly independent with all generators of $\mathbb{L}_1$ and we define a liken $\mathbb{L}_2 = (x_{0,2}, x_{1,2}, \ldots)$ as generated by $\mathbb{L}_1$ and a number A_1. We see that in $\mathbb{L}_2$ we have at least one pair of non-separated generators. In this new liken $\mathbb{L}_2$ we find a generator $x_{k_2,2}$ such that $k_2 > k_1$ and $x_{k_2,2} > M_2 = 20$. Next we choose $A_2 \in (x_{k_2,2}, x_{k_2+1,2})$ which is linearly independent with all generators of the liken $\mathbb{L}_2$ and we define a liken $\mathbb{L}_3$ as generated by $\mathbb{L}_2$ and the number A_2. This new liken has at least two pairs of non separated generators. Further we construct in an analogous way a sequence of likens $\mathbb{L}_n = (x_{0,n}, x_{1,n}, \ldots)$, where $\mathbb{L}_n$ is generated by $\mathbb{L}_{n-1}$ and a suitable number A_{n-1}. In this liken we have a pair of non-separated generators $(x_{k_n,n}, x_{k_{n+1},n})$ where $x_{k_n,n} > 10(n-1)$. Using the above properties one may check that the sum

$$\mathbb{L} = \sum_{n=1}^{\infty} \mathbb{L}_n$$

is a liken with uniqueness which has infinitely many pairs of non-separated generators.

Remark 21
Let us observe, that Property 20 is a direct consequence of Theorem 15. Indeed, one may take as $\mathbb{P}$ any subset of $\mathbb{N}$ such that $\mathbb{N} \setminus \mathbb{P}$ is infinite and in $\mathbb{P}$ there are infinitely many pairs of the form $(m, m+1)$ (for example we may take $\mathbb{P} = \{10k, 10k+1\},\ k \in \mathbb{N}$.)

The name of the next property of likens refers to an old philosophical principle. The so-called *Ockham's razor principle* states that *entities should not be multiplied beyond necessity.*

Before we formulate this property for likens, let us establish some notations. Suppose that $\mathbb{L} = (x_m)_0^\infty$ is a liken. For $n \in \mathbb{N}$ we set $\mathbb{L}^{(n)} = \mathbb{L}(x_1, x_2, \ldots, x_n)$, i.e. $\mathbb{L}^{(n)}$ is a liken generated by all elements not greater than x_n, which is clearly a sub-liken of $\mathbb{L}$. Let us observe that $\mathbb{L}^{(n)} = \mathbb{L}(a_1, a_2, \ldots, a_k)$ where $(a_1, a_2, \ldots, a_k)$ are all indecomposable elements of the liken $\mathbb{L}$ such that $a_i \leq x_n$.

Let

$$z(x_n) = z_n = \min\{x : \ x \in \mathbb{L}^{(n)}, \ x > x_n\}. \tag{2}$$

Property (OR)

We will say that a liken $\mathbb{L}$ has *the Ockham's razor property* if

$$\mathrm{supp}(x_n) \cap \mathrm{supp}(z_n) = \emptyset \Longrightarrow x_{n+1} = z_n.$$

For the sake of explaining the name of *Ockham's razor* let us consider the following: suppose we want to construct a liken $\mathbb{L}$ with the *disjoint support property* and the construction runs recursively. Assume we have constructed x_n and want to construct x_{n+1}. We do the following: we determine the smallest element of the liken generated by the already constructed among bigger than x_n and denote it z_n. If the support of z_n is disjoint with the support of x_n then we take z_n as x_{n+1}. This is just the considered property. And what happens, when $\mathrm{supp}(x_n) \cap \mathrm{supp}(z_n) \neq \emptyset$? Because the "necessity" (for us) is the disjoint support property, then we must "multiply the entities" and set $x_{n+1} = a_{k+1}$. Let us notice here, that

Remark 22

In the notations as above, if $x_{n+1} = a_{k+1}$ then necessarily $x_{n+2} = z_n$.

Indeed, we have $x_n < z_n$, $x_{n+1} = a_{k+1}$, $a_{k+1} \notin \mathbb{L}^n$ and $z_n \in \mathbb{L}^n$ then $x_{n+1} < z_n$. Thus $z_{n+1} = z_n$ and z_n has disjoint support with the support of x_{n+1}. In consequence $x_{n+2} = z_n$. In other words, if x_{n+2} was indecomposable we would have to many "entities".

Property 23

We will say, that a liken $\mathbb{L}((a_k)_1^\infty)$ has *the Bertrand property* when for each $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $x_n \leq a_k \leq x_n + a_1$.

Property 24

We will say, that a liken $\mathbb{L}((a_k)_1^\infty)$ has *the Legendre property* when

$$\lim_{n\to\infty} \frac{\mathrm{card}\{k : \ a_k \leq x_n\}}{n} = 0.$$

All properties: 12-24, (C) and (OR), are true in the liken $\mathbb{N}^*$, so they are consistent. On the other hand it is obvious that the conjunctions of some of the properties on the list above imply other or even all of the others.

In this situation it is natural to ask if there are other likens besides $\mathbb{N}^*$ that have all of properties listed above, or which of these properties characterize the liken of natural numbers with multiplication.

Note that both properties (C) and (OR) are fulfilled in $\mathbb{N}^*$ while $\mathbb{N}^{**}$ has property (C) and does not have the property (OR). Indeed, in this case we have (in multiplicative model): $x_1 = 3$, $x_2 = 5$, $z_2 = 9$ and $x_3 = 7$. Hence concavity do not imply the Ockham's razor property. On the other hand, the property (C) implies the disjoint support property. Indeed, suppose that $x_{k+1} = x_p + a_i$ and $x_k = x_q + a_i$. Hence $\delta(x_k) = x_{k+1} - x_k = x_p - x_q \geq x_{q+1} - x_q = \delta(x_q)$. But this is impossible, since in concave likens $q < k$ implies $\delta(x_q) > \delta(x_k)$.

4. The main theorem

In this section we are going to prove the main theorem, which gives a characterization of the liken $\mathbb{N}^*$ in the space of all likens. Suppose, that $\overrightarrow{a} = (a_k)_1^\infty$ is a sequence of positive real numbers generating a liken $\mathbb{L}(\overrightarrow{a})$ denoted shortly by $\mathbb{L}_a$. (Let us recall, that in this paper, "liken" means "liken with uniqueness"). In these notations we formulate the main result of this paper as follows:

Theorem 25
If the liken $\mathbb{L}_a$ is concave and has the Ockham's razor property, then it is isomorphic to the liken $\mathbb{N}^*$.

First we will make a number of observations, which will be used in the proof.

4.1. Multiplicative notation

Definition 1 of a liken determines, that a liken $\mathbb{L} = (x)_{n=0}^\infty$ is an increasing sequence of non-negative real numbers closed under addition in $\mathbb{R}$. Consider a new sequence defined by the formula $\widehat{x}_n = \exp(x_{n-1})$ for $n = 1, 2, \ldots$. This sequence $\widehat{\mathbb{L}} = (\widehat{x}_n)_1^\infty$ is a strictly increasing sequence of positive real numbers closed with respect to the multiplication in $\mathbb{R}$ and obviously

$$\widehat{x_p + x_q} = \widehat{x}_p \cdot \widehat{x}_q.$$

We may say, that $\widehat{\mathbb{L}}$ is the same liken as $\mathbb{L}$, but we write "$\cdot$" instead of "$+$". The number 0 is replaced by 1 and indices go from 1 to $+\infty$. In particular, the liken $\mathbb{N}$ is transformed to the multiplicative liken $(e^{n-1})_{n=1}^\infty$. Conversely, if we have a liken $\widehat{\mathbb{L}} = (\widehat{x}_n)_{n=1}^\infty$ with the multiplicative notation, than the sequence $(x_n)_0^\infty$ defined by the formula $x_n = \ln(\widehat{x}_{n+1})$ for $n = 0, 1, \ldots$, is a liken with additive notation. In particular, the liken $\mathbb{N}^*$ where $\widehat{x}_1 = 1$, $\widehat{x}_2 = 2$ etc. is transformed to the liken $x_0 = \ln(1), x_1 = \ln(2), x_2 = \ln(3) \ldots$. In consequence, if in an additive liken $\mathbb{L}$ we consider the gaps $\delta_k = x_{k+1} - x_k$ then in the multiplicative version we use the fractions

$$\widehat{\delta}_k = \frac{\widehat{x}_{k+1}}{\widehat{x}_k},$$

and conversely the quotients are replaced by the differences. Let us agree, that if there is a "hat" above the symbols referring to the liken $\mathbb{L}$ then the formulas refer to the multiplicative model of $\mathbb{L}$.

4.2. The isomorphism exponent

Let us take into account the set $\mathbb{N}_0^{\mathbb{N}}$, called in the sequel *the space of exponents* and let $\mathbb{L}_a = (x_n)_0^\infty$ be a liken. Hence, as we have observed above, the map

$$\Omega_{\mathbb{L}} \colon \mathbb{N}_0^{\mathbb{N}} \ni \overrightarrow{m} \to \langle \overrightarrow{a}, \overrightarrow{m} \rangle \in \mathbb{L}_a$$

is a bijection and an isomorphism of semigroups (monoids).

The inverse map

$$\Omega_{\mathbb{L}}^{-1}\colon \mathbb{L}_a \ni x_n \to \Omega_{\mathbb{L}}^{-1}(x_n) \in \mathbb{N}_0^{\mathbb{N}}$$

is also a bijection and is an isomorphism of semigroups (monoids).

When we have another liken $\mathbb{K}_b = (y_n)_0^\infty$, then we can consider an analogous isomorphisms

$$\Omega_{\mathbb{K}}\colon \mathbb{N}_0^{\mathbb{N}} \ni \overrightarrow{m} \to \langle \overrightarrow{b}, \overrightarrow{m} \rangle \in \mathbb{K}_b,$$

as well as

$$\Omega_{\mathbb{K}}^{-1}\colon \mathbb{K}_b \ni y_n \to \Omega_{\mathbb{K}}^{-1}(y_n) \in \mathbb{N}_0^{\mathbb{N}}.$$

The composition

$$\Psi_{\mathbb{K},\mathbb{L}}\colon \mathbb{K}_b \ni y_n \to \Omega_{\mathbb{L}}(\Omega_{\mathbb{K}}^{-1}(y_n)) \in \mathbb{L}_a$$

is an algebraic isomorphism of the likens $\mathbb{K}_b$ and $\mathbb{L}_a$, which allows us to say that

Theorem 26
Each two infinitely generated likens (with uniqueness) are algebraically isomorphic.

Let us notice, that in the case when the sequences of generators are strictly increasing, then the described isomorphism $\Psi_{\mathbb{K},\mathbb{L}}$ is unique.

Now we take as $\mathbb{K}_b$ the particular liken $\mathbb{N}^* = (\ln(n+1))_0^\infty$, denoted as $(y_n)_0^\infty$ and we consider the analogous isomorphisms $\Omega_{\mathbb{N}^*}$ and $\Omega_{\mathbb{N}^*}^{-1}$. We will write simply Ω, since the lower index is implied by the context.

The composed isomorphism $\Psi_{\mathbb{K},\mathbb{L}}$ in this special case will be denoted simply by Ψ. We have then

$$\Psi\colon \mathbb{N}^* \ni y_n \to \Omega_{\mathbb{L}}(\Omega_{\mathbb{N}^*}^{-1}(y_n)) \in \mathbb{L}_a = \Omega_{\mathbb{L}}(\Omega^{-1}(y_n)) \in \mathbb{L}_a.$$

4.3. The beginning of the inductive proof

As we see, the map Ψ is an algebraic isomorphism of $\mathbb{N}^*$ and $\mathbb{L}_a$. It remains to show, that Ψ is also ordinal. This last assertion will be proved by induction. In fact we want to prove, that for each $n \in \mathbb{N}$ we have $\Psi(y_n) = x_n$. Since clearly $\Psi(y_0) = x_0$ then the induction step is: if $\Psi(y_k) = x_k$ for $k \le n$ then $\Psi(y_{n+1}) = x_{n+1}$. Or, in other words, we must prove the implication:

Theorem 27
If for each $0 \le i < j \le n$ the inequality $x_i < x_j$ is equivalent to the inequality $y_i < y_j$, then $\Psi(y_{n+1}) = x_{n+1}$.

First we shall verify that for small n the function Ψ has the claimed property. Clearly, for $n = 0$ we have $x_0 = 0$ (i.e. $\Psi(y_0) = x_0$), as in each liken. Although, from the formal point of view, this is not necessary, we will check in details that $\Psi(y_k) = x_k$ for a few initial $k \in \mathbb{N}$ in order to see how the properties (C) and (OR) work.

Case $n = 1$**.** It must be $x_1 = a_1$, since x_1 must be indecomposable. Indeed, suppose that $x_1 = u + v$, where $u \in \mathbb{L} \ni v$, $u > 0$ and $v > 0$. Hence $0 < u < x_1$, but this is impossible, since x_1 is next after x_0. In other words $x_1 = a_1$. Hence $\Psi(y_1) = x_1$. Let us observe, that the equality $\Psi(y_1) = x_1$ does not require any additional assumption (i.e. it is true in all likens).

Case $n = 2$. It must be $x_2 = a_2$. Indeed, $z(x_1) = 2a_1$ (the definition of $z(x)$ is in (2) and we see, that the supports of $z(x_1)$ and $x_1 = a_1$ are not disjoint. Hence, by (OR), $x_2 = a_2$. Then, clearly $x_1 = a_1 < a_2 = x_2 < 2a_1$.

Case $n = 3$. We have clearly $x_2 = a_2 < 2a_1 < a_1 + a_2$. Hence $z(x_2) = 2a_1$ is disjoint with $x_2 = a_2$. In consequence $x_3 = 2a_1$. Here we use (OR).

Case $n = 4$. We see, that $x_3 = 2a_1$ is still in $\mathbb{L}^{(2)}$. It is also easy to check, that $z(x_3) = a_1 + a_2$. Since the support of $z(x_3)$ is not disjoint with the support of x_3, then $x_4 = a_3$. Here we use once more the (OR) property.

Case $n = 5$. Clearly $x_4 \in \mathbb{L}^{(3)}$ and $z(x_4) = a_1+a_2$ (this follows from $a_1+a_2 < 2a_3$). We see that $z(x_4)$ has the support disjoint with the support of $x_4 = a_3$. Hence, by (OR), $x_5 = a_1 + a_2$.

Case $n = 6$. Since $2a_2 < 2a_3$ then $z(x_5) \in L^{(2)}$. It is clear, that $3a_1 < 2a_1 + a_2 < a_1 + 2a_2$. Using the property (C) for $n = 2$ we obtain $3a_1 = a_1 + 2a_1 = x_1+x_3 < 2x_2 = 2a_2$. In consequence $z(x_5) = 3a_1$. Since $z(x_5)$ is not disjoint with x_5, then $x_6 = a_4$. Let us observe that here we use the property (C) for the first time, since without this property we cannot obtain the inequality $3a_1 < 2a_2$.

Case $n = 7$. Here, as before, and as we will do later, we may apply a general remark: if $x_n < z(x_n)$ and x_n and $z(x_n)$ are not disjoint, then from (OR) we have: $x_{n+1} = a_{k+1}$ and $x_{n+2} = z(x_n)$. This follows from the inequality $z(x_n) < 2a_{k+1}$. Thus $x_7 = 3a_1$.

Case $n = 8$. It follows from the considerations for $n = 6$ and $n = 7$ that $z(x_7) = 2a_2$, hence $x_8 = 2a_2$.

Case $n = 9$. We are now in $\mathbb{L}^{(4)}$, and we compute $z(x_8)$, which belongs a priori to $\mathbb{L}^{(4)}$. But, we have

$$\begin{aligned} a_1 + a_3 - x_8 &= a_1 + a_3 - 2a_2 = a_1 + 2a_3 - 2a_2 - a_3 \\ &= a_1 + 2x_4 - 2a_2 - a_3 > a_1 + x_3 + x_5 - 2a_2 - a_3 \\ &= a_1 + 2a_1 + a_1 + a_2 - 2a_2 - a_3 = 4a_1 - (a_2 + a_3) \\ &= 2x_3 - (x_2 + x_4) > 0. \end{aligned}$$

Since $a_1+a_3 < 2a_1+a_2$ (because $x_4 < x_5$) and clearly $a_1+a_3 < a_1+a_4$, then $z(x_8) = a_1 + a_3$. Since $z(x_8)$ and $x_8 = 2a_2$ are disjoint, then $x_9 = a_1 + a_3$.

Case $n = 10$. Since $x_4 < x_5$ then $x_9 = a_1+a_3 < 2a_1+a_2$. Clearly $2a_1+a_2 < a_1+a_4$ and $2a_1 + a_2 < a_2 + a_3$. Then $z(x_9) = 2a_1 + a_2$ and hence, $x_{10} = a_5$.

We see, that for $0 \leq n \leq 10$ the map Ψ satisfies the claimed properties on isomorphism of likens.

4.4. The induction step

As we are used to the multiplicative structure of the $\mathbb{N}^*$ semigroup, we will write the proof of the main Theorem 25 in the multiplicative convention of both likens $\widehat{\mathbb{L}}_a$ and $\mathbb{N}^*$. Moreover, the role played by even numbers in $\mathbb{N}^*$ suggest some reformulation of the inductive step. Let us say also, that $\widehat{x}_k$ is even when $\widehat{x}_2|\widehat{x}_k$. Let us recall that ($\widehat{\Psi}$ is the multiplicative version of Ψ defined above)

$$\widehat{\Psi}\colon \mathbb{N}^* \ni n \to \widehat{\Psi}(n) \in \widehat{\mathbb{L}}_a$$

is the (unique) algebraic isomorphism of the considered likens, i.e. for each $i, j \in \mathbb{N}$ we have

$$\widehat{\Psi}(i \cdot j) = \widehat{\Psi}(i) \cdot \widehat{\Psi}(j).$$

Thus to prove, that $\widehat{\Psi}$ is an isomorphism of likens we must prove that $\widehat{\Psi}$ is an order isomorphism, which means, as usually for likens, that for each $i \in \mathbb{N}$ we have: $\widehat{\Psi}(i) = \widehat{x}_i$. So to prove Theorem 25 it is sufficient to prove the following theorem ("even" version of the induction step):

Theorem 28
Suppose, that $n \in \mathbb{N}^*$ and that for each $1 \leq i \leq 2n$ we have $\widehat{\Psi}(i) = \widehat{x}_i$. Then $\widehat{\Psi}(2n+1) = \widehat{x}_{2n+1}$ and $\widehat{\Psi}(2n+2) = \widehat{x}_{2n+2}$.

We will begin by formulating a number of observations.

i) Consider the elements $\widehat{\Psi}(2j)$ for $1 \leq j \leq 2n$. Since $\widehat{\Psi}$ is an algebraic isomorphism, then for each $j \leq 2n$ we have $\widehat{\Psi}(2j) = \widehat{\Psi}(2) \cdot \widehat{\Psi}(j) = \widehat{x}_2 \cdot \widehat{x}_j$. Hence if $j \leq n$ then we can write (by induction hypothesis) $\widehat{x}_2 \cdot \widehat{x}_j = \widehat{x}_{2j}$. In particular $\widehat{x}_2 \cdot \widehat{x}_n = \widehat{x}_{2n}$, but we cannot write *a priori* $\widehat{x}_2 \cdot \widehat{x}_{n+1} = \widehat{x}_{2n+2}$ since this is just one of the conditions to prove. However, all these elements $\widehat{x}_2 \cdot \widehat{x}_j$ are even and are obviously in the liken $\mathbb{L}^{(2n)}$.

ii) Since $\widehat{x}_n < \widehat{x}_{n+1}$ then $\widehat{x}_{2n} = \widehat{x}_2 \cdot \widehat{x}_n < \widehat{x}_2 \cdot \widehat{x}_{n+1}$. But in $\mathbb{L}_a$ we have the disjoint support property, so we must have

$$\widehat{x}_2 \cdot \widehat{x}_n = \widehat{x}_{2n} < \widehat{x}_{2n+1} < \widehat{x}_2 \cdot \widehat{x}_{n+1}.$$

In other words this means, that between $\widehat{x}_{2n}$ and $\widehat{x}_2 \cdot \widehat{x}_{n+1}$ there are some elements of the liken $\mathbb{L}_a$ but we do not now how many of these elements are there, and what they are.

iii) Let us consider the set

$$\mathcal{D} = (\widehat{x}_2 \cdot \widehat{x}_n, \widehat{x}_2 \cdot \widehat{x}_{n+1}) \cap \mathbb{L}^{(2n)}.$$

First we will prove that

Lemma 29
If $2n+1$ is composed, then $\widehat{\Psi}(2n+1) \in (\widehat{x}_{2n}, \widehat{x}_2 \cdot \widehat{x}_{n+1})$.

Proof. Let us assume then that $2n+1 = p \cdot q$. Then clearly $p \geq 3$ and $q \geq 3$ and we have to prove the following inequalities:

$$\widehat{x}_{2n} < \widehat{\Psi}(2n+1)$$

and

$$\widehat{\Psi}(2n+1) < \widehat{x}_2 \cdot \widehat{x}_{n+1}. \tag{3}$$

The first inequality follows directly from the inductive assumption. Indeed, the inductive assumption says, in particular, that

$$\widehat{\Psi}\colon \{1, 2, \ldots, 2n\} \to \{1, \widehat{x}_2, \ldots, \widehat{x}_{2n}\}$$

is a bijection. But since $2n+1 \notin [1, 2, \ldots, 2n]$ then $\widehat{\Psi}(2n+1) \notin \{1, \widehat{x}_2, \ldots, \widehat{x}_{2n}\}$ and in consequence $\widehat{x}_{2n} < \widehat{\Psi}(2n+1)$.

The proof of the second inequality is more complicated. Clearly, we may assume that $p \leq q$ and since $p \cdot q$ is odd then p and q are both odd, and we have the inequality

$$3 \leq p \leq q < n.$$

Indeed, suppose $q \geq n$. Then we have $2n+1 = p \cdot q \geq 3 \cdot n$ which is possible only for $n = 1$ but in our case $n \geq 1$.

Let us denote

$$A = \frac{\widehat{\Psi}(2n+2)}{\widehat{\Psi}(2n+1)}.$$

Our aim is to show that $A > 1$. We have (recall that $\widehat{\Psi}$ is an algebraic isomorphism on the whole $\mathbb{N}^*$ and recall that the quotients corresponds to differences in the additive models).

$$A = \frac{\widehat{\Psi}(2n+2)}{\widehat{\Psi}(2n+1)} = \frac{\widehat{\Psi}(2(n+1))}{\widehat{\Psi}(p \cdot q)} = \frac{\widehat{x}_2 \cdot \widehat{x}_{n+1}}{\widehat{x}_p \cdot \widehat{x}_q}.$$

Let us notice here, that in this moment we cannot write $\widehat{x}_p \cdot \widehat{x}_q = \widehat{x}_{pq}$ since $pq > 2n$. But we know, that p is odd, and then $p+1$ is even and $p+1 \leq n$. Thus $p+1 = 2s \leq n$ and then we have $\widehat{x}_{p+1} = \widehat{x}_2 \cdot \widehat{x}_s$. So we may also write

$$\begin{aligned} A &= \frac{\widehat{\Psi}(2n+2)}{\widehat{\Psi}(2n+1)} = \frac{\widehat{\Psi}(2(n+1))}{\widehat{\Psi}(p \cdot q)} = \frac{\widehat{x}_2 \cdot \widehat{x}_{n+1}}{\widehat{x}_p \cdot \widehat{x}_q} = \frac{\widehat{x}_{p+1}}{\widehat{x}_p} \cdot \frac{\widehat{x}_2 \cdot \widehat{x}_{n+1}}{\widehat{x}_2 \cdot \widehat{x}_s \cdot \widehat{x}_q} \\ &= \frac{\widehat{x}_{p+1}}{\widehat{x}_p} \cdot \frac{\widehat{x}_{n+1}}{\widehat{x}_s \cdot \widehat{x}_q}. \end{aligned}$$

Here is the time to replace $\widehat{x}_s \cdot \widehat{x}_q$ by $\widehat{x}_{sq}$ but for this we must bound sq from above. We have $pq = 2n+1$ and $sq < pq = 2n+1$, hence sq is a natural number satisfying $sq \leq 2n$. This is sufficient for our purposes (for the use the induction hypothesis) although a more detailed analysis allows us to prove that $sq \leq \frac{3n}{2}$. So, by induction hypothesis, we may write $\widehat{x}_s \cdot \widehat{x}_q = \widehat{x}_{sq}$ and in consequence we obtain

$$A = \frac{\widehat{x}_{p+1}}{\widehat{x}_p} \cdot \frac{\widehat{x}_{n+1}}{\widehat{x}_s \cdot \widehat{x}_q} = \frac{\widehat{x}_{p+1}}{\widehat{x}_p} \cdot \frac{\widehat{x}_{n+1}}{\widehat{x}_{sq}}.$$

iv) Here we will need a simple lemma resulting from the concavity property. Suppose, that $\mathbb{L} = (x_n)_{n=0}^{\infty}$ is a liken (in additive convention). For fixed j we can consider the sequence $\delta^j(n) = x_{n+j} - x_n$. It appears, that in concave likens (for each j) such a sequence is also strictly decreasing. First we have

Lemma 30
Let $\mathbb{L} = (x_n)_{n=0}^{\infty}$ be a liken satisfying the concavity property and let p and q be arbitrary positive integers such that $1 \leq p < q$. Then $x_{q-1} - x_{p-1} > x_q - x_p$.

Let us recall the notation $\delta(k) = x_{k+1} - x_k$ and recall that in a concave liken we have $\delta(k+1) < \delta(k)$. Hence

$$\begin{aligned} x_q - x_p &= x_q - x_{q-1} + x_{q-1} - x_{q-2} + \cdots + x_{p+1} - x_p \\ &= \delta(q-1) + \delta(q-2) + \cdots + \delta(p) \\ &< \delta(q-2) + \delta(q-3) + \cdots + \delta(p-1) \\ &= x_{q-1} - x_{q-2} + x_{q-2} - x_{q-3} + \cdots + x_p - x_{p-1} \\ &= x_{q-1} - x_{p-1}. \end{aligned}$$

From this lemma, by induction, we obtain the following inequality: if $1 \leq p < q$ and $k \leq p$ then $x_{q-k} - x_{p-k} > x_q - x_p$.

The same, but in multiplicative notation, may be formulated as follows.

Lemma 31
Let us suppose, that $\widehat{x}_p, \widehat{x}_q$ are two elements of a concave liken $\mathbb{L} = (\widehat{x}_n)_{n=0}^{\infty}$ (in multiplicative convention), and $1 \leq k < p < q$. Then

$$\frac{\widehat{x}_p}{\widehat{x}_q} > \frac{\widehat{x}_{p-k}}{\widehat{x}_{q-k}}.$$

v) Now we return to the bounding from below of the quantity A. Our aim is to prove that $A > 1$. We have proved that

$$A = \frac{\widehat{x}_{p+1}}{\widehat{x}_p} \cdot \frac{\widehat{x}_{n+1}}{\widehat{x}_s \cdot \widehat{x}_q} = \frac{\widehat{x}_{p+1}}{\widehat{x}_p} \cdot \frac{\widehat{x}_{n+1}}{\widehat{x}_{sq}}.$$

The inequality $A > 1$ is evident when $n+1 \geq sq$, then assume that $n+1 < sq$. By Lemma 31 we have

$$A = \frac{\widehat{x}_{p+1}}{\widehat{x}_p} \cdot \frac{\widehat{x}_{n+1}}{\widehat{x}_{sq}} > \frac{\widehat{x}_{p+1}}{\widehat{x}_p} \cdot \frac{\widehat{x}_{n+1-s}}{\widehat{x}_{sq-s}}.$$

As we have observed above, we have $n + 1 - s < 2n$ and $sq - s < 2n$ and additionally we will check that

$$\frac{n+1-s}{sq-s} = \frac{p}{p+1}.$$

Indeed, we have the sequence of equivalent equalities

$$\begin{aligned}\frac{n+1-s}{sq-s} = \frac{p}{p+1} &\Leftrightarrow (p+1)(n+1-s) = p(sq-s)\\ &\Leftrightarrow 2s(n+1-s) = ps(q-1)\\ &\Leftrightarrow 2(n+1-s) = p(q-1)\\ &\Leftrightarrow 2n+2-2s = pq-p\\ &\Leftrightarrow 2n+2-p-1 = 2n+1-p.\end{aligned}$$

The last equivalence is true since we assumed that $pq = 2n+1$ and $p+1 = 2s$. Hence there exists a number $t \in \mathbb{N}$ such that $n+1-s = tp$ and $sq-s = t(p+1)$. From the inductive assumption we have

$$A = \frac{\widehat{x}_{p+1}}{\widehat{x}_p} \cdot \frac{\widehat{x}_{n+1}}{\widehat{x}_{sq}} > \frac{\widehat{x}_{p+1}}{\widehat{x}_p} \cdot \frac{\widehat{x}_{n+1-s}}{\widehat{x}_{sq-s}} = \frac{\widehat{x}_{p+1}}{\widehat{x}_p} \cdot \frac{\widehat{x}_{tp}}{\widehat{x}_{t(p+1)}} = \frac{\widehat{x}_{p+1}}{\widehat{x}_p} \cdot \frac{\widehat{x}_t \cdot \widehat{x}_p}{\widehat{x}_t \cdot \widehat{x}_{p+1}} = 1.$$

This ends the proof of (3) and at the same time proof of Lemma 29.

vi) Consider now the situation, when between $\widehat{x}_{2n}$ and $\widehat{x}_2 \cdot \widehat{x}_{n+1}$ there are no elements of the liken $\mathbb{L}^{(2n)}$, i.e. the set $\mathcal{D}$ is empty. In this case $z_{2n} = \widehat{x}_2 \cdot \widehat{x}_{n+1}$. The razor property implies then that $\widehat{x}_{2n+1} = a_{k+1}$. But in this case $2n+1$ cannot be composed, since, when $2n+1$ is composed, then $\widehat{\Psi}(2n+1)$ is in $\mathbb{L}^{(2n)}$, and, as we have proved above

$$\widehat{\Psi}(2n+1) \in (\widehat{x}_{2n}, \widehat{x}_2 \cdot \widehat{x}_{n+1}),$$

contrary to our assumption. Hence $2n+1 = p_{k+1}$ (p_{k+1} is the $(k+1)$-th prime in $\mathbb{N}^*$) and $a_{k+1} = \widehat{x}_{2n+1}$, and we see that in this case $\Psi(2n+1) = \widehat{x}_{2n+1}$.

vii) Summarizing, we have proved, that the element $\widehat{\Psi}(2n+1)$ is always in the interval $(\widehat{x}_{2n}, \widehat{x}_2 \cdot \widehat{x}_{n+1})$. Hence to end the proof of the main Theorem 25 it is enough to show, that in the interval $(\widehat{x}_{2n}, \widehat{x}_2 \cdot \widehat{x}_{n+1})$ there are no other elements of sub-liken $\mathbb{L}^{(2n)}$ besides, possibly, $\widehat{\Psi}(2n+1)$.

viii) Suppose that there exists an element $\widehat{x}$ such that $\widehat{x} \in \widehat{\mathbb{L}}^{(2n)} \cap (\widehat{x}_{2n}, \widehat{x}_2 \cdot \widehat{x}_{n+1})$ and $\widehat{x} \neq \widehat{\Psi}(2n+1)$. Since $\widehat{x} \in \widehat{\mathbb{L}}^{(2n)}$ then there exist two natural numbers r and s, such that $r \leq 2n$, $s \leq 2n$, $\widehat{x} = \widehat{x}_r \cdot \widehat{x}_s$ and $\widehat{x}_{2n} < \widehat{x}_r \cdot \widehat{x}_s < \widehat{x}_2 \cdot \widehat{x}_{n+1}$. Since $\widehat{x} \neq \widehat{\Psi}(2n+1)$ then $r \cdot s > 2n+1$ and since $\widehat{x}_{2n} < \widehat{x}_r \cdot \widehat{x}_s < \widehat{x}_2 \cdot \widehat{x}_{n+1}$ then both r and s are odd. Indeed, if for example $r = 2m$ then $\widehat{x_r} = \widehat{x}_2 \cdot \widehat{x}_m$ and in consequence $\widehat{x}_n < \widehat{x}_m \cdot \widehat{x}_s < \widehat{x}_{n+1}$, what is impossible. Clearly, we can assume that $r \leq s$ and observe, that in fact we have the inequalities

$$3 \leq r \leq s < n.$$

Indeed, since $r > 1$ and r is odd, we have $r \geq 3$, so we must show that $s < n$. Suppose that $s \geq n$. But $s \leq 2n$ and $\widehat{\Psi}$ is increasing in the interval $[1, 2n]$ (induction), thus $\widehat{x}_s \geq \widehat{x}_n$ and in consequence

$$\widehat{x}_2 \cdot \widehat{x}_{n+1} > \widehat{x}_r \cdot \widehat{x}_s > \widehat{x}_3 \cdot \widehat{x}_n.$$

This gives the inequality (in the multiplicative convention)

$$\widehat{x}_2 \cdot \widehat{x}_{n+1} > \widehat{x}_3 \cdot \widehat{x}_n$$

and this gives (in additive convention) the inequality

$$x_{n+1} - x_n > x_2 - x_1,$$

what is impossible in concave likens. Since $n > r \geq 3$ and r is odd, then $r-1$ is even and we can put $r-1 = 2t$. In consequence we have

$$\widehat{x}_2 \cdot \widehat{x}_t \cdot \widehat{x}_s = \widehat{x}_{r-1} \cdot \widehat{x}_s < \widehat{x}_r \cdot \widehat{x}_s < \widehat{x}_2 \cdot, \widehat{x}_{n+1}$$

which gives the inequality

$$\widehat{x}_t \cdot \widehat{x}_s < \widehat{x}_{n+1}.$$

Since we are in the interval $[1, \widehat{x}_2, \widehat{x}_3, \ldots, \widehat{x}_{2n}]$, and $\widehat{\Psi}^{-1}$ is increasing then $t \cdot s < n+1$.

ix) The end of our reasoning is similar as above. We know, that we may set $rs = 2m+1$ where $m \geq n+1$. Let us denote

$$B = \frac{\widehat{\Psi}(2m+1)}{\widehat{\Psi}(2m)}. \tag{4}$$

Our aim is to prove that $B > 1$. We have

$$\begin{aligned} B &= \frac{\widehat{\Psi}(2m+1)}{\widehat{\Psi}(2m)} = \frac{\widehat{\Psi}(r \cdot s)}{\widehat{\Psi}(2m)} = \frac{\widehat{x}_r \cdot \widehat{x}_s}{\widehat{x}_2 \cdot \widehat{x}_m} = \frac{\widehat{x}_r}{\widehat{x}_{r-1}} \cdot \frac{\widehat{x}_{r-1} \cdot \widehat{x}_s}{\widehat{x}_2 \cdot \widehat{x}_m} \\ &= \frac{\widehat{x}_r}{\widehat{x}_{r-1}} \cdot \frac{\widehat{x}_2 \cdot \widehat{x}_t \cdot \widehat{x}_s}{\widehat{x}_2 \cdot \widehat{x}_m} = \frac{\widehat{x}_r}{\widehat{x}_{r-1}} \cdot \frac{\widehat{x}_{ts}}{\widehat{x}_m} > \frac{\widehat{x}_r}{\widehat{x}_{r-1}} \cdot \frac{\widehat{x}_{ts-t}}{\widehat{x}_{m-t}}. \end{aligned}$$

By a similar argument as before, we check that $ts - t = w(r-1)$ and $m - t = wr$. Since $\widehat{\Psi}$ is an algebraic isomorphism in the whole $\mathbb{N}^*$, we have $\widehat{x}_{ts-t} = \widehat{x}_w \cdot \widehat{x}_{r-1}$ and $\widehat{x}_{m-t} = \widehat{x}_w \cdot \widehat{x}_r$.

Let us observe some inequalities. Since we have proved that $st < n+1$ then $ts - t \leq n$ and hence $w \cdot (r-1) \leq n$ and by induction hypothesis, we have

$$\widehat{x}_{ts-t} = \widehat{x}_{w(r-1)} = \widehat{x}_w \cdot \widehat{x}_{r-1}.$$

We must also bound $m - t$ from above. We have $rs > 2m$. Thus

$$(r-1+1)s > 2m \quad \text{and} \quad (r-1)s + s > 2m.$$

But $(r-1) = 2t$ then $2ts + s > 2m$. We have proved that $s < n$ and $ts < n+1$. In consequence $2m < 2ts + s < 2n + 2 + n = 3n + 2$. Hence $m - t < 2n$ and we can use the induction hypothesis for $m - t = wr$. Hence

$$B = \frac{\widehat{\Psi}(2m+1)}{\widehat{\Psi}(2m)} > \frac{\widehat{x}_r}{\widehat{x}_{r-1}} \cdot \frac{\widehat{x}_{ts-t}}{\widehat{x}_{m-t}} = \frac{\widehat{x}_r}{\widehat{x}_{r-1}} \cdot \frac{\widehat{x}_{w(r-1)}}{\widehat{x}_{wr}} = \frac{\widehat{x}_r}{\widehat{x}_{r-1}} \cdot \frac{\widehat{x}_w \cdot \widehat{x}_{r-1}}{\widehat{x}_w \cdot \widehat{x}_r} = 1.$$

Finally we obtain

$$\widehat{x}_2 \cdot \widehat{x}_{n+1} > \widehat{x}_r \cdot \widehat{x}_s = \widehat{\Psi}(2m+1) > \widehat{\Psi}(2m) = \widehat{x}_2 \cdot \widehat{x}_m.$$

In consequence, $\widehat{x}_{n+1} > \widehat{x}_m$. This means that $\widehat{x}_m \in [1, \widehat{x}_2, \dots, \widehat{x}_{2n}]$, so we may use the induction hypothesis and we obtain $n+1 > m$. But we know, that $m \geq n+1$ and this contradiction ends the proof of the inductive step, and at the same time, the proof of the main theorem.

5. Some additional remarks

Remark 32

As we have observed, the space of likens is big, but there are only a few examples of likens which could be described as "suitable for counting". A natural method of obtaining such kind of examples is to choose a subset $K \subset \mathbb{N}^*$ and consider the sub-semigroup $\mathbb{L}(K)$ generated by K (i.e. the smallest semigroup containing the set K) which is ordered by the order inherited from $\mathbb{N}^*$. Hence we obtain a liken, a sub-liken of $\mathbb{N}^*$. In particular we may consider only the likens generated by the subsets of the set of prime numbers. Even this family, small compared to the family of all likens, is nevertheless rich, since it contains a continuum of non-isomorphic likens. Theorem 25 shows, that only one of these likens is concave and has the razor property.

Remark 33

It is commonly known, that the Cauchy functional equation of the type $f(x \cdot y) = f(x) + f(y)$ has many "bad" solutions and only one (up to a constant factor) "good" solution, if we claim f to be continuous (or monotone, or locally bounded etc.) and this solution is the logarithmic function. It follows from Theorem 25 that the condition of convexity for likens together with the razor property may be considered as a kind of condition guaranteeing the uniqueness of the logarithmic function.

Acknowledgement. The author would like to express his gratitude to the knowledgeable referees, whose remarks helped much to improve the presentation of the paper. Special thanks go to the referee who proposed the proof of Theorem 15.

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Received: May 9, 2022; final version: October 5, 2022;
available online: November 28, 2022.