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Deductive systems of pseudo-M algebras

Abstract. The class of pseudo-M algebras contains pseudo-BCK, pseudo-BCI, pseudo-BCH, pseudo-BE, pseudo-CI algebras and many other algebras of logic. In this paper, the notion of deductive system in a pseudo-M algebra is introduced and its elementary properties are investigated. Closed deductive systems are defined and studied. The homomorphic properties of (closed) deductive systems are provided. The concepts of translation deductive systems and R-congruences in pseudo-M algebras are introduced and investigated. It is shown that there is a bijection between closed translation deductive systems and R-congruences. Finally, the construction of quotient algebra $\mathcal{A}/D$ of a pseudo-M algebra $\mathcal{A}$ via a translation deductive system D of $\mathcal{A}$ is given.

1. Introduction

Y. Imai and K. Iséki [11, 15] introduced BCK and BCI algebras as algebras connected with some logics. Q. P. Hu and X. Li [10] defined BCH algebras, which are a generalization of BCI algebras (hence also a generalization of BCK algebras). H. S. Kim and Y. H. Kim introduced in [17] the concept of BE-algebras. It is known that BCK algebras are contained in the class of BE algebras. In 2009, B. L. Meng [20] defined CI algebras as a common generalization of BE and BCH algebras.

In 2001, G. Georgescu and A. Iorgulescu [8] defined pseudo-BCK algebras as a non-commutative extension of BCK algebras. In 2008, W. Dudek and Y. B. Jun [4] introduced pseudo-BCI algebras as a natural generalization of BCI algebras and of pseudo-BCK algebras. Next, pseudo-BE algebras were introduced in 2013

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by R. A. Borzooei et al. [2], as a non-commutative extension of BE algebras. A. Walendziak [28] introduced in 2015 the class of pseudo-BCH algebras. Recently, A. Rezaei et al. [25] defined pseudo-CI algebras as a common generalization of pseudo-BE and pseudo-BCH algebras. All of the algebras mentioned above are contained in the class of pseudo-M algebras introduced by A. Iorgulescu in the book [14] from 2018.

Deductive systems of algebras of logic are an important algebraic notion. From the logical point of view, deductive systems correspond to those sets of formulas which are closed under the inference rule modus ponens. Note that deductive systems are the same as ideals considered among others in the papers [23, 16] on BCK algebras, [9, 24] on BCI algebras, and [3, 29] on BCH and pseudo-BCH algebras (in all these papers, the notation with $*$ and 0 is used). Note also that in the theory of BE algebras and in the theory of CI algebras, deductive systems coincide with filters (see for example [21, 27] and [22, 1]).

In this paper, we introduce the notion of deductive system in a pseudo-M algebra and investigate its elementary properties. We define and study closed deductive systems. We describe deductive systems of direct products of pseudo-M algebras and provide the homomorphic property of (closed) deductive systems. We introduce and investigate the concepts of translation deductive systems and R-congruences in pseudo-M algebras. We prove that the lattice of closed translation deductive systems of a pseudo-RM algebra $\mathcal{A}$ with the property (pD) is isomorphic to the lattice of R-congruences on $\mathcal{A}$. Finally, we give the construction of quotient algebra $\mathcal{A}/D$ of a pseudo-RM algebra $\mathcal{A}$ via a translation deductive system D of $\mathcal{A}$ and obtain the fundamental homomorphism theorem.

The motivation for this work consists algebraic arguments; namely, the pseudo-M algebras are a generalization of various pseudo-algebras. As another motivation of this study we mention the possible applications in the theory of pseudo-BCK, pseudo-BCI, pseudo-BCH, pseudo-BE and pseudo-CI algebras.

2. On pseudo-M algebras

Let $\mathcal{A} = (A, \rightarrow, \rightsquigarrow, 1)$ be an algebra of type $(2, 2, 0)$. We say that $\mathcal{A}$ is a *pseudo-M algebra* (more precisely, *left-pseudo-M algebra*, see [14]) if it verifies the axioms:

(IdEq) $x \rightarrow y = 1 \Leftrightarrow x \rightsquigarrow y = 1$,

(pM) $1 \rightarrow x = x = 1 \rightsquigarrow x$.

A natural binary relation $\leq$ can always be defined on A by: for all $x, y \in A$,

(pEq) $x \leq y \Leftrightarrow x \rightarrow y = 1$ ($\Leftrightarrow x \rightsquigarrow y = 1$).

We consider the following list of properties (cf. [14]) that can be satisfied by a pseudo-M algebra $\mathcal{A}$:

(An) (Antisymmetry) ($x \leq y$ and $y \leq x$)$\Rightarrow x = y$,

(pBB) $y \rightarrow z \leq (z \rightarrow x) \rightsquigarrow (y \rightarrow x)$, $\quad y \rightsquigarrow z \leq (z \rightsquigarrow x) \rightarrow (y \rightsquigarrow x)$, (pB) $y \to z \le (x \to y) \to (x \to z), \quad y \rightsquigarrow z \le (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)$,

(pD) $x \le (x \to y) \rightsquigarrow y, \quad x \le (x \rightsquigarrow y) \to y$,

(pEx) (Exchange property) $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$,

(pC) $x \rightsquigarrow (y \to z) \le y \to (x \rightsquigarrow z), \quad x \to (y \rightsquigarrow z) \le y \rightsquigarrow (x \to z)$,

(L) (Last element) $x \le 1$,

(Re) (Reflexivity) $x \le x$,

(Tr) (Transitivity) ($x \le y$ and $y \le z$)$\Rightarrow x \le z$,

(p*) $x \le y \Rightarrow (z \to x \le z \to y$ and $z \rightsquigarrow x \le z \rightsquigarrow y)$,

(p**) $x \le y \Rightarrow (y \to z \le x \to z$ and $y \rightsquigarrow z \le x \rightsquigarrow z)$.

Proposition 2.1
If an algebra $\mathcal{A}$ satisfies (pM) *and*

(pD') $x \rightsquigarrow ((x \to y) \rightsquigarrow y) = 1, \quad x \to ((x \rightsquigarrow y) \to y) = 1$,

then $\mathcal{A}$ also satisfies (IdEq) *and* (Re)*.*

Proof. Straightforward.

Lemma 2.2 (see Proposition 2.1.8 and Corollary 2.1.11 of [14])
Let $\mathcal{A}$ be a pseudo-M algebra. Then the following hold:

(i) (Re) + (pEx) + (p*) $\Rightarrow$ (pBB),

(ii) (pBB) $\Rightarrow$ (p**) $\Rightarrow$ (Tr),

(iii) (pBB) $\Rightarrow$ (Re), (pC), (pD),

(iv) (Re) + (pEx) $\Rightarrow$ (pD),

(v) (pBB) $\Rightarrow$ (pB) $\Rightarrow$ (p*),

(vi) (pBB) + (An) $\Rightarrow$ (pEx).

Definition 2.3 (see [14])
An algebra $\mathcal{A} = (A, \to, \rightsquigarrow, 1)$ is called:

1. a *pseudo-RM algebra* if it is a pseudo-M algebra verifying (Re);
2. a *pseudo-ML algebra* if it is a pseudo-M algebra verifying (L);
3. a *pseudo-RML algebra* if it is a pseudo-RM algebra verifying (L);
4. a *pseudo-CI algebra* (or pseudo-RME algebra) if it is a pseudo-RM algebra verifying (pEx);
5. a *pseudo-BE algebra* if it is a pseudo-CI algebra verifying (L);
6. a *pseudo-BCH algebra* if it is a pseudo-CI algebra verifying (An);
7. a *pseudo-BCI algebra* if it is a pseudo-BCH algebra verifying (p*);
8. a *pseudo-BCK algebra* if it is a pseudo-BCI algebra verifying (L).

Denote by **pM**, **pML**, **pRM**, **pRML**, **pCI**, **pBE**, **pBCH**, **pBCI**, **pBCK** the classes of pseudo-M, pseudo-ML, pseudo-RM, pseudo-RML, pseudo-CI, pseudo-BE, pseudo-BCH, pseudo-BCI, pseudo-BCK algebras, respectively.

We say that a pseudo-M algebra $\mathcal{A} = (A, \to, \rightsquigarrow, 1)$ is:

9. a *pseudo-BB algebra* if it verifies (pBB);
10. a *pseudo-CI* algebra* if it verifies (Re), (pEx) and (p*) – that is, it is a pseudo-CI algebra with (p*);
11. a *pseudo-BE* algebra* if it verifies (Re), (pEx), (L) and (p*) – that is, it is a pseudo-BE algebra with (p*).

Denote by **pBB**, **pCI***, **pBE*** the classes of pseudo-BB, pseudo-CI*, pseudo-BE* algebras, respectively.

From Lemma 2.2 (i) we see that **pCI*** $\subset$ **pBB**. The interrelationships between the classes of algebras mentioned before are visualized in Fig. 1. (An arrow indicates proper inclusion, that is, if **X** and **Y** are classes of algebras, then **X** $\longrightarrow$ **Y** means **X** $\subset$ **Y**.)

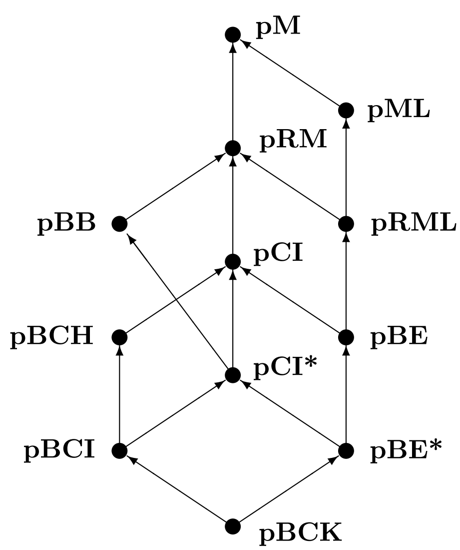

Fig. 1

A pseudo-M algebra $(A, \to, \rightsquigarrow, 1)$ is called a *M algebra* if $\to\, = \,\rightsquigarrow$. More formally, an algebra $(A, \to, 1)$ of type $(2, 0)$ is said to be a *M algebra* if it satisfies

(M) $1 \to x = x$.

Similarly, if X $\in$ {M, RM, RML, CI, BE, BCH, BCI, BCK}, then a pseudo-X algebra $(A, \to, \to, 1)$ is called a *X algebra.* These algebras and many new generalizations of BCI or of BCK algebras can be found in [13].

In pseudo-RM, pseudo-RML, pseudo-CI and pseudo-BE algebras, $\leq$ is a reflexive relation; in pseudo-BB, pseudo-CI* and pseudo-BE* algebras, it is reflexive and transitive by Lemma 2.2 (i), (ii). In pseudo-BCH algebras, by definition, $\leq$ is reflexive and antisymmetric. It is known that $\leq$ is an order relation in pseudo-BCI and pseudo-BCK algebras.

Proposition 2.4
The class of pseudo-BB algebras is a variety of algebras of type $(2,2,0)$ *with the associated set* $\{\to,\rightsquigarrow,1\}$ *of operation symbols, defined by identities* (pM) *and*

(pBB') $(y\to z)\rightsquigarrow((z\to x)\rightsquigarrow(y\to x))=1,\quad (y\rightsquigarrow z)\to((z\rightsquigarrow x)\to(y\rightsquigarrow x))=1.$

Proof. Let $\mathcal{A}=(A,\to,\rightsquigarrow,1)$ be an algebra of type $(2,2,0)$ satisfying (pM) and (pBB'). Let $x,y,z\in A$. By (pBB') and (pM), $1=(1\to z)\rightsquigarrow((z\to x)\rightsquigarrow(1\to x))=z\rightsquigarrow((z\to x)\rightsquigarrow x)$ and $1=(1\rightsquigarrow z)\to((z\rightsquigarrow x)\to(1\rightsquigarrow x))=z\to((z\rightsquigarrow x)\to x)$, that is, (pD') holds. From Proposition 2.1 we conclude that (IdEq) also holds. Therefore, $\mathcal{A}$ is a pseudo-M algebra satisfying (pBB). Thus $\mathcal{A}$ is a pseudo-BB algebra.

Example 2.5
Let $A=\{a,b,1\}$ and consider the following implications:

$\to_1$	a	b	1
a	a	1	a
b	1	1	1
1	a	b	1

,

$\rightsquigarrow_1$	a	b	1
a	b	1	b
b	1	1	1
1	a	b	1

;

$\to_2$	a	b	1
a	a	1	1
b	b	1	1
1	a	b	1

,

$\rightsquigarrow_2$	a	b	1
a	b	1	1
b	a	1	1
1	a	b	1

.

Then, $\mathcal{A}_1=(A,\to_1,\rightsquigarrow_1,1)$ is a pseudo-M algebra not verifying (Re) and (L) for a, (An) for a, b (pD) for a, 1; $\mathcal{A}_2=(A,\to_2,\rightsquigarrow_2,1)$ is a pseudo-ML algebra not verifying (Re) for a, (pD) for 1, a (hence not (pBB)).

Example 2.6
Consider the set $A=\{a,b,1\}$ and the following tables of implications:

$\to_1$	a	b	1
a	1	1	1
b	b	1	b
1	a	b	1

,

$\rightsquigarrow_1$	a	b	1
a	1	1	1
b	a	1	a
1	a	b	1

;

$\to_2$	a	b	1
a	1	1	a
b	1	1	b
1	a	b	1

,

$\rightsquigarrow_2$	a	b	1
a	1	1	b
b	1	1	b
1	a	b	1

.

Let $\mathcal{A}_1=(A,\to_1,\rightsquigarrow_1,1)$ and $\mathcal{A}_2=(A,\to_2,\rightsquigarrow_2,1)$. Then, $\mathcal{A}_1$ is a pseudo-RM algebra and $\mathcal{A}_2$ is a pseudo-BB algebra. They do not verify (L) for $x=b$, (pEx) for $(x, y, z) = (a, b, a)$. Moreover, $\mathcal{A}_1$ does not verify (pD) for $(x, y) = (b, a)$ and (pBB) for $(x, y, z) = (1, a, b)$.

Example 2.7
Let $A = \{a, b, c, 1\}$ and consider the following implications:

$\to$	a	b	c	1
a	1	1	b	1
b	1	1	c	1
c	a	b	1	1
1	a	b	c	1

,

$\rightsquigarrow$	a	b	c	1
a	1	1	b	1
b	1	1	c	1
c	b	c	1	1
1	a	b	c	1

.

Then, $\mathcal{A} = (A, \to, \rightsquigarrow, 1)$ is a pseudo-RML algebra. It does not satisfy (An), (pBB), (pEx), (pD).

For examples of pseudo-CI, pseudo-CI*, pseudo-BE and pseudo-BE* algebras we refer to [2, 25], of pseudo-BCH algebras to [28], of pseudo-BCI algebras to [7, 18] and of pseudo-BCK algebras to [12, 19]. Note that Iorgulescu's pre-pseudo-BCI algebras are our pseudo-CI* algebras and Iorgulescu's pre-pseudo-BCK algebras are our pseudo-BE* algebras.

3. Deductive systems

In this section, we define the notion of deductive system of a pseudo-M algebra and give some of its properties.

Let $\mathcal{A} = (A, \to, \rightsquigarrow, 1)$ be a pseudo-M algebra. We say that a subset D of A is a *deductive system* of $\mathcal{A}$ if it satisfies:

(D1) $1 \in D$,

(D2) for all $x, y \in A$, if $x \in D$ and $x \to y \in D$, then $y \in D$,

(D3) for all $x, y \in A$, if $x \in D$ and $x \rightsquigarrow y \in D$, then $y \in D$.

By $\mathrm{DS}(\mathcal{A})$ we denote the set of all deductive systems of $\mathcal{A}$. It is obvious that $\{1\}$, $A \in \mathrm{DS}(\mathcal{A})$. The algebras from Example 2.6 have only two deductive systems: $\{1\}$, A.

Remark 3.1
Let $\mathcal{A}_2$ be the pseudo-RML algebra from Example 2.7. The set $D = \{1, c\}$ satisfies (D2), but it does not satisfy (D3).

Example 3.2
Let $A = \{a, b, c, d, 1\}$ and $\to, \rightsquigarrow$ be defined by the following tables:

$\to$	a	b	c	d	1
a	1	1	c	c	c
b	1	1	c	c	c
c	a	b	1	1	c
d	a	b	b	1	c
1	a	b	c	d	1

,

$\rightsquigarrow$	a	b	c	d	1
a	1	1	c	c	d
b	1	1	c	c	d
c	a	b	1	1	d
d	a	b	b	1	d
1	a	b	c	d	1

.

By routine calculation, $\mathcal{A} = (A, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-RM algebra, without (pD). It is easy to see that $\mathrm{DS}(\mathcal{A}) = \{\{1\}, \{1, d\}, \{1, a, b\}, \{1, c, d\}, A\}$.

Proposition 3.3
Let $\mathcal{A}$ be a pseudo-M algebra with the property (pD) *and let D be a subset of A containing* 1. *Then the following conditions are equivalent:*

(a) *D is a deductive system of $\mathcal{A}$,*

(b) *D verifies* (D2),

(c) *D verifies* (D3).

Proof. By definition, (a) implies (b). To prove that (b) implies (a), let D be a subset of A verifying (D1) and (D2). Suppose that $x \rightsquigarrow y \in D$ and $x \in D$. By (pD), $x \leq (x \rightsquigarrow y) \rightarrow y$. Then $x \rightarrow ((x \rightsquigarrow y) \rightarrow y) = 1 \in D$, and we conclude from (D2) that $(x \rightsquigarrow y) \rightarrow y \in D$, hence that $y \in D$. Therefore, D verifies (D3), and consequently D is a deductive system. Thus (a) $\Leftrightarrow$ (b). The proof of the equivalence (a) $\Leftrightarrow$ (c) is analogous.

Corollary 3.4
If a pseudo-RM algebra $\mathcal{A}$ satisfies (pBB) *or* (pEx), *then a subset D of A is a deductive system of $\mathcal{A}$ if and only if it satisfies* (D1) *and* (D2), *or equivalently, if and only if it satisfies* (D1) *and* (D3).

Corollary 3.5
In pseudo-CI algebras (hence also in pseudo-BE, pseudo-BCH, pseudo-BCI, pseudo-BCK algebras), a subset D is a deductive system if and only if it satisfies (D1) *and* (D2), *or equivalently,* (D1) *and* (D3).

Proposition 3.6
Let D be a deductive system of a pseudo-M algebra $\mathcal{A}$. Then, for any $x, y \in A$, if $x \leq y$ and $x \in D$, then $y \in D$.

Proof. Straightforward.

Theorem 3.7
Let $\mathcal{A}_i = (A_i, \rightarrow_i, \rightsquigarrow_i, 1_i)$, $i \in I$, be an indexed family of pseudo-RML algebras and $\mathcal{A} := \prod_{i \in I} \mathcal{A}_i$ be the direct product of these algebras. Then

(i) *if D_i is a deductive system of $\mathcal{A}_i$ for $i \in I$, then $\prod_{i \in I} D_i$ is a deductive system of $\mathcal{A}$,*

(ii) *if D is a deductive system of $\mathcal{A}$, then $D_i := \pi_i(D)$, where π_i is the i-th projection of $\mathcal{A}$ onto $\mathcal{A}_i$, is a deductive system of $\mathcal{A}_i$, and $D \subseteq \prod_{i \in I} D_i$.*

Proof. (i) The first part of the assertion is obvious.

(ii) Obviously, $1_i \in D_i$. Let $x_i, y_i \in A_i$ and suppose that $x_i \rightarrow_i y_i \in D_i$ and $x_i \in D_i$. Define $x, y \in A$ by

$$x(j) = \begin{cases} x_i, & \text{if } j = i, \\ 1_j, & \text{if } j \neq i \end{cases}$$

and

$$y(j)=\begin{cases} y_i, & \text{if } j=i,\\ 1_j, & \text{if } j\neq i\end{cases}$$

for any $j \in I$. Then $x(i) \in D_i$ and $x(i) \to_i y(i) \in D_i$. Hence, there exists an element $z \in D$ such that $\pi_i(z) = x(i) \to_i y(i)$. We have

$$(x\to y)(j)=\begin{cases} z(i), & \text{if } j=i.\\ 1_j, & \text{if } j\neq i.\end{cases}$$

Since $z(j) \leq 1_j$, for $j \neq i$, and $\mathcal{A}_i$ satisfies (Re), we deduce that $z \leq x \to y$. As D is a deductive system and $z \in D$, we conclude that $x \to y \in D$. Let now $t \in D$ be such that $\pi_i(t) = x_i$. Clearly, $t \leq x$. Hence $x \in D$, because $t \in D$. Consequently, $y \in D$, and therefore $y_i = \pi_i(y) \in \pi_i(D) = D_i$. Thus D_i satisfies (D2). Similarly, D_i also satisfies (D3). This means that D_i is a deductive system of $\mathcal{A}_i$.

Finally, it is easy to see that $D \subseteq \prod_{i\in I} D_i$.

Definition 3.8
A deductive system D of a pseudo-M algebra $\mathcal{A}$ is said to be *closed* if $x \to 1 \in D$ and $x \rightsquigarrow 1 \in D$ for every $x \in D$.

By definition, we have

Proposition 3.9
In pseudo-RML algebras, hence in pseudo-BE, pseudo-BCK algebras, every deductive system is closed.

Theorem 3.10
Let $\mathcal{A}$ be a pseudo-RM algebra with the property (pBB) *or* (pEx). *Then a deductive system D of $\mathcal{A}$ is closed if and only if D is a subalgebra of $\mathcal{A}$.*

Proof. Suppose that D is a closed deductive system of $\mathcal{A}$ and let $x, y \in D$. Observe that

$$x \to 1 \leq y \rightsquigarrow (x \to y). \tag{3.1}$$

Indeed, if $\mathcal{A}$ satisfies (pBB), then $x \to 1 \leq (1 \to y) \rightsquigarrow (x \to y) = y \rightsquigarrow (x \to y)$. If (pEx) holds, then $x \to 1 = x \to (y \rightsquigarrow y) = y \rightsquigarrow (x \to y)$. Therefore, $\mathcal{A}$ satisfies (3.1). Since D is closed, $x \to 1 \in D$. We conclude from (3.1) that $y \rightsquigarrow (x \to y) \in D$, hence that $x \to y \in D$. Similarly, $x \rightsquigarrow y \in D$. Conversely, if D is a subalgebra of $\mathcal{A}$, then $x \in D$ and $1 \in D$ imply $x \to 1 \in D$.

Corollary 3.11
Let $\mathcal{A}$ be a pseudo-CI algebra. Then every closed deductive system of $\mathcal{A}$ is a subalgebra.

Remark 3.12
From Corollary 3.11 we have Proposition 3.5 of [22] and Theorem 4.16 of [28].

Remark 3.13
Let $\mathcal{A}$ be the pseudo-M algebra from Example 3.2. Then $\{1, c, d\}$ is a closed deductive system of $\mathcal{A}$ but it is not a subalgebra.

For any $x, y \in A$ and $n \in \mathbb{N}_0$, we define $x \to^n y$ inductively as follows:

$$x \to^0 y = y \quad \text{and} \quad x \to^{n+1} y = x \to (x \to^n y) \quad \text{for } n \in \mathbb{N}_0,$$

We define $x \rightsquigarrow^n y$ in the same way.

In [28], it is proved that every deductive system of a finite pseudo-BCH algebra is closed. Similarly, we obtain

Proposition 3.14
Every deductive system of a finite pseudo-BB algebra is closed.

Remark 3.15
From Proposition 3.14 we get Theorem 4.13 [5] for pseudo-BCI algebras.

Remark 3.16
A pseudo-BB algebra in which every deductive system is closed does not have to be finite; for example, we take any infinite pseudo-BCK algebra. Observe also that the assumption of finiteness in Proposition 3.14 cannot be dropped, see example below.

Example 3.17
Let M be the set of all matrices of the form $\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$, where x and y are rational numbers such that $x > 0$. We define the binary operations $\to$ and $\rightsquigarrow$ on M by

$$X \to Y = YX^{-1} \quad \text{and} \quad X \rightsquigarrow Y = X^{-1}Y$$

for all $X, Y \in M$. Then $\mathfrak{M} = (M, \to, \rightsquigarrow, E)$, where $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, is a pseudo-BB algebra. Let $C = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. It is easy to see that the set $D = \{C^n : \ n \in \mathbb{N} \cup \{0\}\}$ is a deductive system of $\mathfrak{M}$. Observe that D is not closed. Indeed, $C \to E = EC^{-1} = C^{-1} \notin D$.

For any pseudo-M algebra $\mathcal{A}$, we set

$$K(\mathcal{A}) = \{x \in A : \ x \leq 1\}.$$

Proposition 3.18
If a pseudo-M algebra $\mathcal{A}$ satisfies (p**)*, then* $\mathrm{K}(\mathcal{A})$ *is a closed deductive system of $\mathcal{A}$.*

Proof. Obviously, $1 \in \mathrm{K}(\mathcal{A})$. Let x, $x \to y \in \mathrm{K}(\mathcal{A})$. Then $x \leq 1$ and $x \to y \leq 1$. From (pM) and (p**) we obtain $y = 1 \to y \leq x \to y \leq 1$. Hence $y \leq 1$ ($\mathcal{A}$ satisfies (Tr) by Lemma 2.2 (ii)), that is, $y \in \mathrm{K}(\mathcal{A})$. Similarly we prove that if x, $x \rightsquigarrow y \in \mathrm{K}(\mathcal{A})$, then $y \in \mathrm{K}(\mathcal{A})$. Thus $\mathrm{K}(\mathcal{A})$ is a deductive system of $\mathcal{A}$. It is clear that if $x \in \mathrm{K}(\mathcal{A})$, then $x \to 1 \in \mathrm{K}(\mathcal{A})$. Therefore, $\mathrm{K}(\mathcal{A})$ is closed.

Remark 3.19
From Lemma 2.2 we conclude that Proposition 3.18 holds for pseudo-BB, pseudo-CI*, pseudo-BE*, pseudo-BCI, pseudo-BCK algebras.

Remark 3.20
Example 3.21 shows that the converse of Proposition 3.18 does not hold in general.

Example 3.21
Let $A = \{a, b, c, 1\}$ and consider the following implications:

$\to$	a	b	c	1
a	a	b	c	a
b	a	b	c	b
c	a	b	c	1
1	a	b	c	1

,

$\rightsquigarrow$	a	b	c	1
a	a	c	b	a
b	a	b	c	b
c	a	b	c	1
1	a	b	c	1

.

Then, $\mathcal{A} = (A, \to, \rightsquigarrow, 1)$ is a pseudo-M algebra. We have $\mathrm{K}(\mathcal{A}) = \{1, c\}$. It is clear that $\mathrm{K}(\mathcal{A})$ is a closed deductive system of $\mathcal{A}$, whereas $\mathcal{A}$ does not satisfy (p**), since $c \leq 1$, but $1 \to b = b \nleq b = c \to b$.

Proposition 3.22
Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be pseudo-M algebras. Then:

(i) $\mathrm{K}(\mathcal{A}_1 \times \mathcal{A}_2) = \mathrm{K}(\mathcal{A}_1) \times \mathrm{K}(\mathcal{A}_2)$,

(ii) *if $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy* (Re) *and* (L)*, then* $\mathrm{DS}(\mathcal{A}_1 \times \mathcal{A}_2) = \mathrm{DS}(\mathcal{A}_1) \times \mathrm{DS}(\mathcal{A}_2)$.

Proof. (i) This is immediate from definitions.

(ii) Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ and $D \in \mathrm{DS}(\mathcal{A})$. From Theorem 3.7 (ii) it follows that $D \subseteq D_1 \times D_2$, where $D_1 = \pi_1(D)$, $D_2 = \pi_2(D)$. Let $a \in D_1$ and $b \in D_2$. There are $c \in A_2$ and $d \in A_1$ such that $(a, c), (d, b) \in D$. Since $(a, c) \leq (a, 1)$ and $(d, b) \leq (1, b)$, we conclude that $(a, 1), (1, b) \in D$. Observe that $(a, b) \in D$. Indeed, we have $(a, 1) \to (a, b) = (1, b)$ and $(a, 1), (1, b) \in D$. From this $(a, b) \in D$. Therefore $D = D_1 \times D_2 \in \mathrm{DS}(\mathcal{A}_1) \times \mathrm{DS}(\mathcal{A}_2)$.

Conversely, let $D = D_1 \times D_2$, where $D_1 \in \mathrm{DS}(\mathcal{A}_1)$ and $D_2 \in \mathrm{DS}(\mathcal{A}_2)$. By Theorem 3.7 (i), D is a deductive system of $\mathcal{A}$.

The following two propositions give the homomorphic properties of deductive systems.

Proposition 3.23
Let $\mathcal{A}$ and $\mathcal{B}$ be pseudo-M algebras. If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a homomorphism and D is a (closed) deductive system of $\mathcal{B}$, then the inverse image $\varphi^{-1}(D)$ of D is a (closed) deductive system of $\mathcal{A}$.

Proof. Straightforward.

Proposition 3.24
Let $\mathcal{A} = (A, \to, \rightsquigarrow, 1_A)$ and $\mathcal{B} = (B, \to, \rightsquigarrow, 1_B)$ be pseudo-RM algebras and $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism. If D is a (closed) deductive system of $\mathcal{A}$ containing $\ker \varphi := \varphi^{-1}(\{1_B\})$*, then $\varphi(D)$ is a (closed) deductive system of $\mathcal{B}$.*

Proof. Since $1_A \in D$, we have $1_B = \varphi(1_A) \in \varphi(D)$. Let $x, y \in B$ and suppose that $x, x \to y \in \varphi(D)$. Then there are $a, b \in D$ and $c \in A$ such that $x = \varphi(a)$, $x \to y = \varphi(b)$ and $y = \varphi(c)$. We get $x \to y = \varphi(a) \to \varphi(c) = \varphi(b)$, so $\varphi(a \to c) =$

$\varphi(b)$. Hence $b \to (a \to c) \in \ker\varphi$, and consequently $b \to (a \to c) \in D$. By the definition of deductive system, $c \in D$. Therefore, $y = \varphi(c) \in \varphi(D)$. Similarly, if $x, x \rightsquigarrow y \in \varphi(D)$, then $y \in \varphi(D)$. This means that $\varphi(D)$ is a deductive system of $\mathcal{B}$. Moreover, if D is closed, then $\varphi(D)$ is also closed.

Remark 3.25
Propositions 3.23 and 3.24 imply Propositions 4.13 and 4.14 of [28] on ideals of pseudo-BCH algebras, Theorem 14 of [2] on filters of pseudo-BE algebras.

The set $\mathrm{DS}(\mathcal{A})$ of all deductive systems of a pseudo-M algebra $\mathcal{A}$ (partially ordered by set-inclusion) is a complete lattice where infima coincide with set-theoretical intersections. Hence, for any $X \subseteq A$ there exists the least deductive system containing X. Denote it by $D(X)$ and call it the *deductive system generated by X*.

Proposition 3.26
Let $\mathcal{A}$ be a pseudo-BB algebra and let $\emptyset \neq X \subseteq A$ and

$$\begin{aligned} Y &= \{x \in A :\ a_1 \to (\cdots \to (a_n \to x)\cdots) = 1 \textit{ for some } a_1, \ldots, a_n \in X,\ n \in \mathbb{N}\} \\ &= \{x \in A :\ a_1 \rightsquigarrow (\cdots \rightsquigarrow (a_n \rightsquigarrow x)\cdots) = 1 \textit{ for some } a_1, \ldots, a_n \in X,\ n \in \mathbb{N}\}. \end{aligned}$$

Then $D(X) = Y \cup \{1\}$ and $D(\emptyset) = \{1\}$.

Proof. By (pC) and (p*), we get

$$a_1 \to (\cdots \to (a_n \to x)\cdots) = 1 \Leftrightarrow a_n \rightsquigarrow (\cdots \rightsquigarrow (a_1 \rightsquigarrow x)\cdots) = 1$$

for any $x, a_1, \ldots, a_n \in A$ and $n \in \mathbb{N}$. Then, it suffices to prove that $Y \cup \{1\}$ is the least deductive system containing X. Obviously, $1 \in Y \cup \{1\}$. Suppose that $x, x \to y \in Y \cup \{1\}$. If $x = 1$, then $y = 1 \to y \in Y \cup \{1\}$. Let $x \in Y$. By the definition of Y, there exist $a_1, \ldots, a_n \in X$ such that

$$a_1 \rightsquigarrow (\cdots \rightsquigarrow (a_n \rightsquigarrow x)\cdots) = 1. \tag{3.2}$$

We shall consider two cases:

Case **1.** $x \to y = 1$. Then $x \leq y$. Since $\mathcal{A}$ satisfies (p*), we have $a_n \rightsquigarrow x \leq a_n \rightsquigarrow y$. Repeating this way $n-1$ times and noticing (3.2) we obtain

$$1 = a_1 \rightsquigarrow (\cdots \rightsquigarrow (a_n \rightsquigarrow x)\cdots) \leq a_1 \rightsquigarrow (\cdots \rightsquigarrow (a_n \rightsquigarrow y)\cdots),$$

that is, $a_1 \rightsquigarrow (\cdots \rightsquigarrow (a_n \rightsquigarrow y)\cdots) = 1$. Hence $y \in Y \subseteq Y \cup \{1\}$.

Case **2.** $x \to y \in Y$. Therefore, there are $b_1, \ldots, b_m \in X$ $(m \in \mathbb{N})$ such that

$$b_1 \rightsquigarrow (\cdots \rightsquigarrow (b_m \rightsquigarrow (x \to y))\cdots) = 1. \tag{3.3}$$

Applying (pC) we get

$$b_m \rightsquigarrow (x \to y) \leq x \to (b_m \rightsquigarrow y).$$

Repeating this, by (p*), (pC) and (Tr), we obtain

$$b_1 \rightsquigarrow (\cdots \rightsquigarrow (b_m \rightsquigarrow (x \to y))\cdots) \leq x \to (b_1 \rightsquigarrow (\cdots \rightsquigarrow (b_m \rightsquigarrow y)\cdots)).$$

Hence, using (3.3), we have $1 = x \to (b_1 \rightsquigarrow (\cdots \rightsquigarrow (b_m \rightsquigarrow y)\cdots))$, and consequently, $x \leq b_1 \rightsquigarrow (\cdots \rightsquigarrow (b_m \rightsquigarrow y)\cdots)$. Then, by (p*),

$$\begin{aligned} 1 &= a_1 \rightsquigarrow (\cdots \rightsquigarrow (a_n \rightsquigarrow x)\cdots) \\ &\leq a_1 \rightsquigarrow (\cdots \rightsquigarrow (a_n \rightsquigarrow (b_1 \rightsquigarrow (\cdots \rightsquigarrow (b_m \rightsquigarrow y)\cdots)))\cdots). \end{aligned}$$

Thus $y \in Y \subseteq Y \cup \{1\}$. Therefore, $Y \cup \{1\} \in \mathrm{DS}(\mathcal{A})$. It is obvious that $X \subseteq Y \subseteq Y \cup \{1\}$.

To prove that $Y \cup \{1\}$ is the least deductive system containing X, let $D \in \mathrm{DS}(\mathcal{A})$ and $D \supseteq X$. For any $x \in Y \cup \{1\}$, if $x = 1$, then $x \in D$; otherwise, there are $a_1, \ldots, a_n \in X$ such that (3.2) holds. Since $X \subseteq D$, we have $a_1, \ldots, a_n \in D$. By the definition of a deductive system, from (3.2) we see that $x \in D$. Hence $Y \cup \{1\} \subseteq D$. Thus $D(X) = Y \cup \{1\}$.

Moreover, it is easily seen that $D(\emptyset) = \{1\}$.

Remark 3.27
In particular, from Theorem 3.26 we obtain Lemma 2.1.4 of [19], Corollary 5.7 of [20], Proposition 5.23 of [25], Proposition 3.7 of [6].

By Proposition 3.26, the mapping $X \to D(X)$ is an algebraic closure operator on the power set of A, that is, for every $X \subseteq A$,

$$D(X) = \bigcup\{D(X_0) : \ X_0 \text{ is a finite subset of } X\}.$$

Hence we obtain

Theorem 3.28
For any pseudo-BB algebra $\mathcal{A}$, $\mathrm{DS}(\mathcal{A})$ forms an algebraic lattice whose compact elements are precisely the finitely generated deductive systems.

Remark 3.29
Theorem 3.28 was proved for deductive systems of pseudo-BCK algebras by J. Kühr [19] and of pseudo-BCI algebras by G. Dymek [6].

4. Translation deductive systems and congruences

In this section we first study the notion of compatibility of deductive systems and then we introduce translation deductive systems and R-congruences. We say that a deductive system D of a pseudo-M algebra is *compatible* if, for all $x, y \in A$,

$$x \to y \in D \Leftrightarrow x \rightsquigarrow y \in D.$$

We use $\mathrm{DS}_{\mathrm{com}}(\mathcal{A})$ to denote the set of all compatible deductive systems of a pseudo-M algebra $\mathcal{A}$. It is easy to see that $\{1\}, A \in \mathrm{DS}_{\mathrm{com}}(\mathcal{A})$.

Example 4.1
Let $\mathcal{A}$ be the pseudo-M algebra from Example 3.2. The deductive system $D_1 = \{1, c, d\}$ is compatible, while $D_2 = \{1, d\}$ is not compatible.

Proposition 4.2
Let $\mathcal{A} = (A, \to, \rightsquigarrow, 1_A)$ *and* $\mathcal{B} = (B, \to, \rightsquigarrow, 1_B)$ *be pseudo-M algebras and* $\varphi\colon \mathcal{A} \to \mathcal{B}$ *be a homomorphism from* $\mathcal{A}$ *into* $\mathcal{B}$*. Then* $\ker\varphi = \varphi^{-1}(\{1_B\})$ *is a closed compatible deductive system of* $\mathcal{A}$*.*

Proof. Straightforward.

We have the following characterization of compatibility of deductive systems.

Theorem 4.3
Let $\mathcal{A}$ *be a pseudo-M algebra satisfying* (pD) *and* (p**)*. Let* $D \in \mathrm{DS}(\mathcal{A})$*. The following statements are equivalent:*

(a) *D is compatible,*

(b) *$(d \to x) \to x \in D$ and $(d \rightsquigarrow x) \rightsquigarrow x \in D$, for all $d \in D$ and $x \in A$.*

Proof. (a) $\Rightarrow$ (b): Let $d \in D$ and $x \in A$. From (pD) it follows that $d \le (d \to x) \rightsquigarrow x$ and $d \le (d \rightsquigarrow x) \to x$. By Proposition 3.6, $(d \to x) \rightsquigarrow x \in D$ and $(d \rightsquigarrow x) \to x \in D$. Since D is compatible, we conclude that $(d \to x) \to x \in D$ and $(d \rightsquigarrow x) \rightsquigarrow x \in D$.

(b) $\Rightarrow$ (a): Let $x \to y \in D$. By assumption, $((x \to y) \rightsquigarrow y) \rightsquigarrow y \in D$. Applying (pD), we get $x \le (x \to y) \rightsquigarrow y$, and hence, by (p**), $((x \to y) \rightsquigarrow y) \rightsquigarrow y \le x \rightsquigarrow y$. Consequently, $x \rightsquigarrow y \in D$. Similarly, $x \rightsquigarrow y \in D$ yields $x \to y \in D$. Thus D is compatible.

Proposition 4.4
Let $\mathcal{A}$ *be a pseudo-RM algebra with* (pD) *and* (p**)*. Then* $\mathrm{K}(\mathcal{A})$ *is a closed compatible deductive system.*

Proof. By Proposition 3.18, $\mathrm{K}(\mathcal{A})$ is a closed deductive system of $\mathcal{A}$. Let $d \in \mathrm{K}(\mathcal{A})$ and $x \in A$. Then $d \le 1$. Applying (p**), we have $1 \to x \le d \to x$. Hence $(d \to x) \to x \le (1 \to x) \to x = 1$, and consequently $(d \to x) \to x \in \mathrm{K}(\mathcal{A})$. Similarly, $(d \rightsquigarrow x) \rightsquigarrow x \in \mathrm{K}(\mathcal{A})$. From Theorem 4.3 we conclude that $\mathrm{K}(\mathcal{A}) \in \mathrm{DS}_{\mathrm{com}}(\mathcal{A})$.

Proposition 4.5
Let $\mathcal{A} = (A, \to, \rightsquigarrow, 1_A)$ *and* $\mathcal{B} = (B, \to, \rightsquigarrow, 1_B)$ *be pseudo-RM algebras and* $\varphi\colon \mathcal{A} \to \mathcal{B}$ *be a surjective homomorphism. If D is a compatible deductive system of $\mathcal{A}$ containing* $\ker\varphi$*, then $\varphi(D)$ is a compatible deductive system of $\mathcal{B}$.*

Proof. Let $D \in \mathrm{DS}(\mathcal{A})$ and $D \supseteq \ker\varphi$. By Proposition 3.24, $\varphi(D)$ is a deductive system of $\mathcal{B}$. Suppose that D is compatible. To prove that $\varphi(D)$is compatible, let $x, y \in B$ and $x \to y \in \varphi(D)$. Then there are $a, b \in A$ and $d \in D$ such that $x = \varphi(a)$, $y = \varphi(b)$ and $x \to y = \varphi(d)$. Hence $\varphi(a \to b) = \varphi(d)$, and consequently $d \to (a \to b) \in \ker\varphi \subseteq D$. Therefore $a \to b \in D$ and using the compatibility of D, we have $a \rightsquigarrow b \in D$. Then $x \rightsquigarrow y = \varphi(a \rightsquigarrow b) \in \varphi(D)$. Similarly, $x \rightsquigarrow y \in \varphi(D)$ yields $x \to y \in \varphi(D)$. Thus $\varphi(D)$ is a compatible deductive system.

In [26], E. H. Roch et al. introduced the notion of translation ideal in BCH algebras. In [29, 30], we defined translation ideals and translation deductive systems in pseudo-BCH and RM algebras, respectively. Similarly, we have

Definition 4.6
A compatible deductive system D of a pseudo-M algebra $\mathcal{A}$ is said to be a *translation deductive system* (*t-deductive system*, for short) if it satisfies the following condition: for all $x, y, z \in A$,

$$x \to y, y \to x \in D \Rightarrow (x \diamond z) \to (y \diamond z),\ (z \diamond x) \to (z \diamond y) \in D, \qquad (4.1)$$

where $\diamond \in \{\to, \rightsquigarrow\}$.

Denote by $\mathrm{T}(\mathcal{A})$, $\mathrm{T}_{\mathrm{cl}}(\mathcal{A})$ the set of all t-deductive systems and closed t-deductive systems of $\mathcal{A}$. Obviously, $A \in \mathrm{T}_{\mathrm{cl}}(\mathcal{A})$. But $\{1\}$ is not a translation deductive system, in general (see examples below).

Example 4.7
Consider the pseudo-RM algebra $\mathcal{A}_1$ from Example 2.5. We have $a \to_1 b, b \to_1 a \in \{1\}$, but $(a \to_1 1) \to_1 (b \to_1 1) = a \to_1 1 = a \notin \{1\}$. Therefore, $\{1\}$ is not a translation deductive system of $\mathcal{A}_1$.

Example 4.8
Let $A = \{a, b, c, d, 1\}$. Define binary operations $\to$ and $\rightsquigarrow$ on A by the following tables:

$\to$	a	b	c	d	1
a	1	1	a	a	a
b	1	1	b	b	b
c	a	b	c	d	1
d	a	b	c	1	c
1	a	b	c	d	1

$\rightsquigarrow$	a	b	c	d	1
a	1	1	a	a	a
b	1	1	b	b	b
c	a	b	d	d	1
d	a	b	c	1	c
1	a	b	c	d	1

Then $\mathcal{A} = (A, \to, \rightsquigarrow, 1)$ is a pseudo-M algebra (not verifying (An), (Re), (L), (pD), (pBB), (pEx)). It is evident that $\mathrm{DS}(\mathcal{A}) = \{\{1\}, \{1, c\}, \{1, d\}, \{1, c, d\}, A\}$. The set $\{1\}$ is not a translation deductive system, because $d \to d = 1 \in \{1\}$ but $(d \to c) \to (d \to c) = c \to c = c \notin \{1\}$. The deductive systems $\{1, c\}$ and $\{1, d\}$ are not compatible, since $c \to c = c$ but $c \rightsquigarrow c = d$. We have $\mathrm{T}(\mathcal{A}) = \mathrm{T}_{\mathrm{cl}}(\mathcal{A}) = \{\{1, c, d\}, A\}$.

Proposition 4.9
Let $\mathcal{A}$ be a pseudo-RM algebra satisfying (An). *Then* $\{1\} \in \mathrm{T}(\mathcal{A})$.

Proof. Straightforward.

Theorem 4.10
If $\mathcal{A}$ is a pseudo-BB algebra, then $\mathrm{DS}_{\mathrm{com}}(\mathcal{A}) = \mathrm{T}(\mathcal{A})$.

Proof. By Lemma 2.2 (v), $\mathcal{A}$ satisfies (pB). Let D be a compatible deductive system of $\mathcal{A}$ and let $x, y, z \in A$. Suppose that $x \to y, y \to x \in D$. Using the compatibility of D, we get $x \rightsquigarrow y, y \rightsquigarrow x \in D$. By (pBB), $y \to x \leq (x \to z) \rightsquigarrow (y \to z)$ and $y \rightsquigarrow x \leq (x \rightsquigarrow z) \to (y \rightsquigarrow z)$. Since $y \to x \in D$ and $y \rightsquigarrow x \in D$, we see that $(x \to z) \to (y \to z) \in D$ and $(x \rightsquigarrow z) \to (y \rightsquigarrow z) \in D$. From (pB) it follows that $x \to y \leq (z \to x) \to (z \to y)$ and $x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$. We have $(z \to x) \to (z \to y), (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y) \in D$, because $x \to y, x \rightsquigarrow y \in D$. Thus D is a translation deductive system of $\mathcal{A}$.

Corollary 4.11
In pseudo-CI, pseudo-BE*, pseudo-BCI, pseudo-BCK algebras we have*

$$\mathrm{DS}_{\mathrm{com}}(\mathcal{A}) = \mathrm{T}(\mathcal{A}).$$

Proposition 4.12
If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a homomorphism of pseudo-M algebras and $D \in \mathrm{T}(\mathcal{B})$, then $\varphi^{-1}(D) \in \mathrm{T}(\mathcal{A})$.

Proof. Straightforward.

Proposition 4.13
Let $\mathcal{A} = (A; \to, \rightsquigarrow, 1_A)$and $\mathcal{B} = (B; \to, \rightsquigarrow, 1_B)$ be pseudo-RM algebras and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism. If D is a translation deductive system of $\mathcal{A}$ containing $\ker\varphi$, then $\varphi(D)$ is a translation deductive system of $\mathcal{B}$.

Proof. By Proposition 4.5, $\varphi(D)$ is a compatible deductive system of $\mathcal{B}$. Let $x, y, z \in B$. Since φ is onto, there exist $a, b, c \in A$ such that $x = \varphi(a)$, $y = \varphi(b)$ and $z = \varphi(c)$. To prove that $\varphi(D) \in \mathrm{T}(\mathcal{B})$, let $x \to y, y \to x \in \varphi(D)$. Then $\varphi(a \to b) = \varphi(a) \to \varphi(b) = x \to y \in \varphi(D)$, and similarly, $\varphi(b \to a) \in \varphi(D)$. Hence $\varphi(a \to b) = \varphi(d_1)$ and $\varphi(b \to a) = \varphi(d_2)$ for some $d_1, d_2 \in D$. Therefore, $\varphi(d_1 \to (a \to b)) = \varphi(d_2 \to (b \to a)) = 1_B$, and consequently

$$d_1 \to (a \to b), d_2 \to (b \to a) \in \ker\varphi \subseteq D.$$

It follows that $d_1 \to (a \to b), d_2 \to (b \to a) \in D$. By definition, $a \to b, b \to a \in D$. Since D is a translation deductive system,

$$(a \diamond c) \to (b \diamond c), (c \diamond a) \to (c \diamond b) \in D,$$

where $\diamond \in \{\to, \rightsquigarrow\}$. This forces

$$(x \diamond z) \to (y \diamond z) = (\varphi(a) \diamond \varphi(c)) \to (\varphi(b) \diamond \varphi(c)) \in \varphi(D),$$

and similarly, $(z \diamond x) \to (z \diamond y) \in \varphi(D)$. Thus $\varphi(D) \in \mathrm{T}(\mathcal{B})$.

Proposition 4.14
Let $\mathcal{A} = (A, \to, \rightsquigarrow, 1_A)$ and $\mathcal{B} = (B \to, \rightsquigarrow, 1_B)$ be pseudo-RM algebras and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism. If $\mathcal{B}$ satisfies (An)*, then* $\ker\varphi$ *is a closed t-deductive system of $\mathcal{A}$.*

Proof. By Proposition 4.2, $\ker\varphi$ is a closed compatible deductive system of $\mathcal{A}$. Let $\mathcal{B}$ satisfy (An). Then $\{1_B\}$ is a t-deductive system of $\mathcal{B}$ by Proposition 4.9. From Proposition 4.12 we conclude that $\ker\varphi \in \mathrm{T}(\mathcal{A})$. Finally, $\ker\varphi$ is a closed t-deductive system of $\mathcal{A}$.

Let $\mathcal{A}$ be a pseudo-M algebra and θ be an equivalence relation on A. We say that θ is a *congruence* on $\mathcal{A}$ if $x_1\theta y_1$ and $x_2\theta y_2$ imply $x_1 \to x_2\theta y_1 \to y_2$ and $x_1 \rightsquigarrow x_2\theta y_1 \rightsquigarrow y_2$ for all $x_1, x_2, y_1, y_2 \in A$. Let us denote by $\operatorname{Con}\mathcal{A}$ the set of all congruences on $\mathcal{A}$. For $\theta \in \operatorname{Con}\mathcal{A}$ and $x \in A$, we write x/θ for the congruence class containing x, that is, $x/\theta = \{y \in A : y\theta x\}$. Denote $A/\theta = \{x/\theta : x \in A\}$. Set $x/\theta \to' y/\theta = x \to y/\theta$ and $x/\theta \rightsquigarrow' y/\theta = x \rightsquigarrow y/\theta$. The operations $\to'$ and $\rightsquigarrow'$ are well-defined, since θ is a congruence on $\mathcal{A}$.

Lemma 4.15
Let $\mathcal{A}$ be a pseudo-M algebra and $\theta \in \operatorname{Con}\mathcal{A}$. Then $1/\theta$ is a closed deductive system.

Proof. Straightforward.

Lemma 4.16
If $\mathcal{A}$ is a pseudo-M algebra with (pD) *and $\theta \in \operatorname{Con}\mathcal{A}$, then θ satisfies the following condition*

$$x \to y\theta 1 \Leftrightarrow x \rightsquigarrow y\theta 1 \tag{4.2}$$

for all $x, y \in A$.

Proof. Straightforward.

Proposition 4.17
If $\mathcal{A}$ is a pseudo-M algebra satisfying (pD) *and $\theta \in \operatorname{Con}\mathcal{A}$, then:*

(i) *$1/\theta$ is a closed compatible deductive system of $\mathcal{A}$,*

(ii) *$\mathcal{A}/\theta = (A/\theta, \to', \rightsquigarrow', 1/\theta)$ is a pseudo-M algebra with* (pD).

Proof. (i) By Lemma 4.15, $1/\theta$ is a closed deductive system. From Lemma 4.16 it follows that θ satisfies (4.2). Then $1/\theta$ is compatible.

(ii) By (4.2), $\mathcal{A}/\theta$ verifies (IdEq) and, obviously, (pM) and (pD). Therefore, $\mathcal{A}/\theta$ is a pseudo-M algebra with (pD).

Example 4.18
Consider the set $A = \{a, b, c, d, 1\}$ with the following tables of implications:

$\to$	a	b	c	d	1
a	1	b	b	d	1
b	a	1	c	d	1
c	1	1	1	d	1
d	d	d	d	1	1
1	a	b	c	d	1

,

$\rightsquigarrow$	a	b	c	d	1
a	1	b	c	d	1
b	a	1	a	d	1
c	1	1	1	d	1
d	d	d	d	1	1
1	a	b	c	d	1

.

Then $\mathcal{A} = (A, \to, \rightsquigarrow, 1)$ is a pseudo-BE algebra (see [25]). By Lemma 2.2 (iv), it also satisfies (pD). Let $\theta = \{a, b, c, 1\}^2 \cup \{(d, d)\}$. Obviously, $\theta \in \operatorname{Con}\mathcal{A}$. We have $1/\theta = \{a, b, c, 1\}$. It is easy to check that $1/\theta$ is a closed compatible deductive system.

Corollary 4.19
If $\mathcal{A}$ is a pseudo-BB algebra and $\theta \in \operatorname{Con}\mathcal{A}$, then $1/\theta$ is a closed t-deductive system.

Proof. By Lemma 2.2 (iii), $\mathcal{A}$ satisfies (pD). From Proposition 4.17 (i) and Theorem 4.10 we deduce that $1/\theta$ is a closed t-deductive system.

We say that $\theta \in \operatorname{Con}\mathcal{A}$ is an *R-congruence* on a pseudo-M algebra $\mathcal{A}$ if it satisfies the following conditions: for all $x, y \in A$,

(R1) $x \to y\theta 1$ and $y \to x\theta 1$ imply $x\theta y$,

(R2) $x \rightsquigarrow y\theta 1$ and $y \rightsquigarrow x\theta 1$ imply $x\theta y$.

Proposition 4.20
If $\mathcal{A}$ satisfies (pD)*, then* (R1)$\Leftrightarrow$(R2)*.*

Proof. Let (R1) hold. Suppose that $x \rightsquigarrow y\theta 1$ and $y \rightsquigarrow x\theta 1$. Hence $x \to ((x \rightsquigarrow y) \to y)\theta x \to y$, and so $x \to y\theta 1$. Similarly, $y \to x\theta 1$. By (R1), $x\theta y$. Thus (R2) is satisfied. Analogously, (R2) implies (R1).

We will denote by $\operatorname{Con}_{\mathrm{R}}(\mathcal{A})$ the set of all R-congruences on $\mathcal{A}$.

Example 4.21
Let $A = \{a, b, c, 1\}$ and $\to, \rightsquigarrow$ be defined by the following tables:

$\to$	a	b	c	1
a	1	1	c	a
b	1	1	c	1
c	a	b	c	1
1	a	b	c	1

,

$\rightsquigarrow$	a	b	c	1
a	1	1	c	b
b	1	1	c	1
c	a	b	c	1
1	a	b	c	1

.

Then, $\mathcal{A} = (A, \to, \rightsquigarrow, 1)$ is a pseudo-M algebra. Let $\theta = \Delta \cup \{(a,1), (1,a), (b,1), (1,b), (a,b), (b,a)\}$, where $\Delta = \{(a,a), (b,b), (c,c), (1,1)\}$ is the diagonal relation. It is clear that $\Delta, \theta \in \operatorname{Con}\mathcal{A}$. Moreover, θ is an R-congruence on $\mathcal{A}$, while $\Delta \notin \operatorname{Con}_{\mathrm{R}}(\mathcal{A})$.

Proposition 4.22
Let $\mathcal{A}$ be a pseudo-M algebra satisfying (pD) *and $\theta \in \operatorname{Con}_{\mathrm{R}}(\mathcal{A})$. Then $1/\theta$ is a closed t-deductive system of $\mathcal{A}$.*

Proof. By Proposition 4.17 (i), $1/\theta$ is a closed compatible deductive system of $\mathcal{A}$. Let $x \to y \in 1/\theta$ and $y \to x \in 1/\theta$. Then $x \to y\theta 1$ and $y \to x\theta 1$. Since θ satisfies (R1), we have $x\theta y$. Therefore, $x \to z\theta y \to z$ and hence $(x \to z) \to (y \to z)\theta 1$. Thus $(x \to z) \to (y \to z) \in 1/\theta$. Similarly, $(z \to x) \to (z \to y) \in 1/\theta$. Consequently, $1/\theta$ satisfies (4.1) for $\diamond = \to$. Analogously, it satisfies (4.1) for $\diamond = \rightsquigarrow$. Thus $1/\theta$ is a closed t-deductive system of $\mathcal{A}$.

Proposition 4.23
Let $\mathcal{A}$ be a pseudo-M algebra satisfying (pD) *and let $\theta \in \operatorname{Con}\mathcal{A}$. Then the following are equivalent:*

(a) *θ is an R-congruence on $\mathcal{A}$,*

(b) *$\mathcal{A}/\theta$ satisfies* (An).

Proof. (a) $\Rightarrow$ (b): Let θ be an R-congruence on $\mathcal{A}$ and $x, y \in A$. Suppose that $x/\theta \to y/\theta = 1/\theta = y/\theta \to x/\theta$. Then $x \to y/\theta = 1/\theta = y \to x/\theta$. Hence $x \to y\theta 1$ and $y \to x\theta 1$. Since θ satisfies (R1), we have $x\theta y$. Therefore, $x/\theta = y/\theta$, and consequently, (An) holds in $\mathcal{A}/\theta$.

(b) $\Rightarrow$ (a): Let $x, y \in A$. Assume that $x \to y\theta 1$ and $y \to x\theta 1$. Hence $x/\theta \to y/\theta = 1/\theta = y/\theta \to x/\theta$. Since $\mathcal{A}/\theta$ satisfies (An), we get $x/\theta = y/\theta$, that is, $x\theta y$. Thus θ is an R-congruence on $\mathcal{A}$.

We will show that R-congruences are characterized by translation deductive systems. For $D \in \mathrm{T}(\mathcal{A})$, we define

$$x \sim_D y \Leftrightarrow x \to y \in D \quad \text{and} \quad y \to x \in D.$$

Theorem 4.24
Let $\mathcal{A}$ be a pseudo-RM algebra with (pD) *and D be a translation deductive system of $\mathcal{A}$. Then:*

(i) *$\sim_D$ is an R-congruence on $\mathcal{A}$,*

(ii) *$1/{\sim_D} \subseteq D$ is a closed t-deductive system of $\mathcal{A}$,*

(iii) *$1/{\sim_D} = D$ if and only if D is closed,*

(iv) *if $\theta \in \mathrm{Con}_{\mathrm{R}}(\mathcal{A})$, then $\theta = \sim_{1/\theta}$.*

Proof. (i) By (Re), $x \to x = 1 \in D$, that is, $x \sim_D x$ for any $x \in A$. This means that $\sim_D$ is reflexive. From definition, $\sim_D$ is symmetric. To prove that $\sim_D$ is transitive, let $x \sim_D y$ and $y \sim_D z$. Then $x \to y$, $y \to x \in D$ and $y \to z$, $z \to y \in D$. Since D is a t-deductive system, $(y \to z) \to (x \to z) \in D$ and $(z \to y) \to (z \to x) \in D$. Hence, by the definition of deductive system, we get $x \to z \in D$ and $z \to x \in D$. Consequently, $x \sim_D z$, and so $\sim_D$ is transitive. Thus $\sim_D$ is an equivalence relation on A.

Let $x, y, z \in A$ and suppose that $x \sim_D y$. Then $x \to y \in D$ and $y \to x \in D$. Since D is a t-deductive system, $(x \diamond z) \to (y \diamond z)$, $(y \diamond z) \to (x \diamond z) \in D$, where $\diamond \in \{\to, \rightsquigarrow\}$. Thus

$$x \diamond z \sim_D y \diamond z. \tag{4.3}$$

Moreover, $(z \diamond x) \to (z \diamond y) \in D$ and $(z \diamond y) \to (z \diamond x) \in D$. Therefore

$$z \diamond x \sim_D z \diamond y. \tag{4.4}$$

Let now $x \sim_D y$ and $u \sim_D v$. From (4.3) it follows that $x \diamond u \sim_D y \diamond u$. By (4.4), $y \diamond u \sim_D y \diamond v$. Since $\sim_D$ is transitive, we have $x \diamond u \sim_D y \diamond v$. Consequently, $\sim_D$ is an congruence on $\mathcal{A}$.

Observe that $\sim_D$ satisfies (R1) (hence also (R2)). Let $x \to y \sim_D 1$ and $y \to x \sim_D 1$. Then $x \to y = 1 \to (x \to y) \in D$ and $y \to x = 1 \to (y \to x) \in D$. Consequently, $x \sim_D y$. Thus $\sim_D \in \mathrm{Con}_{\mathrm{R}}(\mathcal{A})$.

(ii) From Proposition 4.22 we see that $1/{\sim_D} \subseteq D$ is a closed t-deductive system of $\mathcal{A}$.

(iii) If $1/{\sim_D} = D$, then it is obvious that D is closed. Conversely, assume that D is closed. It is sufficient to show that $D \subseteq 1/{\sim_D}$. Let $x \in D$. Then $1 \to x = x \in D$ and $x \to 1 \in D$, that is, $x \in 1/{\sim_D}$. Therefore, $1/{\sim_D} = D$.

(iv) For an R-congruence θ on $\mathcal{A}$, we have $x \sim_{1/\theta} y \Leftrightarrow x \to y,\ y \to x \in 1/\theta \Leftrightarrow x\theta y$. Thus $\theta = {\sim_{1/\theta}}$.

As usual, a deductive system D of a pseudo-M algebra $\mathcal{A}$ is called the *kernel* of the congruence θ on $\mathcal{A}$ if $D = 1/\theta$.

Theorem 4.25
Every closed translation deductive system of a pseudo-RM algebra $\mathcal{A}$ satisfying (pD) *is the kernel of some R-congruence on $\mathcal{A}$.*

Proof. Let $D \in \mathrm{T_{cl}}(\mathcal{A})$. By Proposition 4.24 (i), $\sim_D \in \mathrm{Con_R}(\mathcal{A})$. Moreover, we have

$$x \in 1/{\sim_D} \Leftrightarrow x \sim_D 1 \Leftrightarrow x \to 1,\ x = 1 \to x \in D \Leftrightarrow x \in D.$$

Therefore, $D = 1/{\sim_D}$.

The sets $\mathrm{T_{cl}}(\mathcal{A})$ and $\mathrm{Con_R}(\mathcal{A})$ partially ordered by set-inclusion are obviously lattices. Moreover, we get

Theorem 4.26
For any pseudo-RM algebra $\mathcal{A}$ with (pD)*, the lattices* $\mathrm{T_{cl}}(\mathcal{A})$ *and* $\mathrm{Con_R}(\mathcal{A})$ *are isomorphic.*

Proof. We consider the function

$$\varphi\colon \theta \to 1/\theta \quad \text{for all } \theta \in \mathrm{Con_R}(\mathcal{A}).$$

By Proposition 4.22, φ maps $\mathrm{Con_R}(\mathcal{A})$ into $\mathrm{T_{cl}}(\mathcal{A})$. Since any closed t-deductive system of $\mathcal{A}$ is the kernel of some R-congruence on $\mathcal{A}$, we conclude that φ is onto $\mathrm{T_{cl}}(\mathcal{A})$. Observe that

$$\theta_1 \subseteq \theta_2 \Leftrightarrow \varphi(\theta_1) \subseteq \varphi(\theta_2) \tag{4.5}$$

for all $\theta_1, \theta_2 \in \mathrm{Con_R}(\mathcal{A})$. If $\theta_1 \subseteq \theta_2$, then clearly $1/\theta_1 \subseteq 1/\theta_2$, that is, $\varphi(\theta_1) \subseteq \varphi(\theta_2)$. Conversely, assume $1/\theta_1 \subseteq 1/\theta_2$. Let $x\theta_1 y$. Then $x \to y\theta_1 1$ and hence $x \to y\theta_2 1$. Similarly, $y \to x\theta_2 1$. Since θ_2 satisfies (R1), we have $x\theta_2 y$. Thus $\theta_1 \subseteq \theta_2$. Consequently, φ maps $\mathrm{Con_R}(\mathcal{A})$ onto $\mathrm{T_{cl}}(\mathcal{A})$ and satisfies (4.5). Therefore $\mathrm{Con_R}(\mathcal{A})$ is isomorphic to $\mathrm{T_{cl}}(\mathcal{A})$.

Example 4.27
Consider the set $A = \{a, b, c, d, e, 1\}$ with the following tables of implications:

$\to$	a	b	c	d	e	1
a	1	1	c	d	e	1
b	a	1	c	d	e	1
c	c	a	1	d	e	1
d	a	b	c	1	1	d
e	a	b	c	1	1	e
1	a	b	c	d	e	1

,

$\rightsquigarrow$	a	b	c	d	e	1
a	1	1	c	d	e	1
b	a	1	c	d	e	1
c	c	b	1	d	e	1
d	a	b	c	1	1	d
e	a	b	c	1	1	e
1	a	b	c	d	e	1

.

Then, $\mathcal{A} = (A, \to, \rightsquigarrow, 1)$ is a pseudo-RM algebra verifying (pD). It is easy to check that $D_1 = \{1\}$, $D_2 = \{1, b\}$, $D_3 = \{1, a, b\}$, $D_4 = \{1, a, b, c\}$, $D_5 = \{1, d, e\}$, $D_6 = \{1, b, d, e\}$, $D_7 = \{1, a, b, d, e\}$, $D_8 = A$ are deductive systems of $\mathcal{A}$. Observe that D_2 is not compatible. Indeed, $c \rightsquigarrow b = b \in D_2$ but $c \to b = a \notin D_2$. Similarly, D_6 is not compatible. D_3 does not satisfy (4.1), since $a \to b, b \to a \in D_3$, but $(c \to a) \to (c \to b) = c \to a = c \notin D_3$. Analogously, D_7 is not a t-deductive system. We have $\mathrm{T}_{\mathrm{cl}}(\mathcal{A}) = \{D_1, D_4, D_5, D_8\}$. From Theorem 4.26 we conclude that the lattice $\mathrm{Con}_{\mathrm{R}}(\mathcal{A})$ is isomorphic to the lattice of Fig. 2, that is, it is the four-element Boolean algebra.

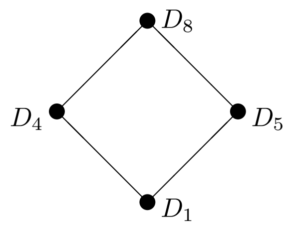

Fig. 2

Let D be a t-deductive system of a pseudo-RM algebra $\mathcal{A}$ satisfying (pD). For $x \in A$, we write $x/D = \{y \in A : \ x \sim_D y\}$. We note that $x \sim_D y$ if and only if $x/D = y/D$, that is,

$$x/D = y/D \Leftrightarrow x \to y,\ y \to x \in D.$$

In particular,

$$x/D = 1/D \Leftrightarrow x = 1 \to x,\ x \to 1 \in D.$$

Denote $A/D = \{x/D : \ x \in A\}$. Set $x/D \to' y/D = x \to y/D$ and $x/D \rightsquigarrow' y/D = x \rightsquigarrow y/D$. The operations $\to'$ and $\rightsquigarrow'$ are well-defined, since $\sim_D$ is a congruence on $\mathcal{A}$.

Proposition 4.28
Let $\mathcal{A}$ be a pseudo-RM algebra with (pD) *and D be a closed t-deductive system of $\mathcal{A}$. Then $\mathcal{A}/D := (A/D; \to', \rightsquigarrow', D)$ is a pseudo-RM algebra satisfying* (An) *and* (pD).

Proof. Theorem 4.24 (i) yields $\theta := \sim_D \in \mathrm{Con}_{\mathrm{R}}(\mathcal{A})$. We have $x/\theta = x/{\sim_D} = x/D$, for all $x \in A$, and $1/\theta = D$, since D is closed. Then $\mathcal{A}/\theta = \mathcal{A}/D$ and, by Propositions 4.17 (ii) and 4.23 (b), $\mathcal{A}/D$ is a pseudo-RM algebra satisfying (An) and (pD).

The algebra $\mathcal{A}/D$ is called the *quotient pseudo-RM algebra* of $\mathcal{A}$ modulo D.

Corollary 4.29
If $\mathcal{A}$ is a pseudo-BB algebra and D is a closed compatible deductive system of $\mathcal{A}$, then $\mathcal{A}/D$ is a pseudo-BCI algebra.

Proof. By Theorem 4.10, $D \in \mathrm{T}_{\mathrm{cl}}(\mathcal{A})$. From Proposition 4.28 we deduce that $\mathcal{A}/D$ is a pseudo-RM algebra with (An). Clearly, $\mathcal{A}/D$ satisfies (pBB) and hence, by Lemma 2.2 (v), (vi), it also satisfies (pEx), (p*). Thus, $\mathcal{A}/D$ is a pseudo-BCI algebra.

Example 4.30
Let $A = \{a, b, c, d, 1\}$ and consider implications: $\to, \rightsquigarrow$ from Example 4.18 and the following implication:

$\twoheadrightarrow$	a	b	c	d	1
a	1	a	a	b	b
b	b	1	1	a	a
c	c	1	1	a	a
d	a	b	c	1	1
1	a	b	c	d	1

.

It can check that $\mathcal{A}_1 = (A, \twoheadrightarrow, \twoheadrightarrow, 1)$ and $\mathcal{A}_2 = (A, \to, \rightsquigarrow, 1)$ verify (Re), (pM) and (pBB). Then $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is a pseudo-BB algebra. It does not verify (An) for $x = (b, c)$, $y = (c, b)$, (L) for $x = (a, a)$, and (pEx) for $x = (a, a)$, $y = (c, a)$, $z = (b, a)$. Let $D = \{1, d\} \times \{1, b, c\}$. It is easy to see that D is a closed compatible deductive system of $\mathcal{A}$. We have

$$\begin{aligned}
\mathbf{a} &:= (b,1)/D = \{(b,1),(b,b),(b,c),(c,1),(c,c),(c,b)\},\\
\mathbf{b} &:= (1,a)/D = \{(1,a),(1,d),(d,a),(d,d)\},\\
\mathbf{c} &:= (b,a)/D = \{(b,a),(b,d),(c,a),(c,d)\},\\
\mathbf{d} &:= (a,b)/D = \{(a,b),(a,c),(a,1)\},\\
\mathbf{e} &:= (a,a)/D = \{(a,a),(a,d)\},\\
\mathbf{1} &:= (1,1)/D = D.
\end{aligned}$$

Set $A/D = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{1}\}$ and $(x_1, y_1)/D \to' (x_2, y_2)/D = (x_1 \twoheadrightarrow x_2, y_1 \to y_2)/D$, $(x_1, y_1)/D \rightsquigarrow' (x_2, y_2)/D = (x_1 \twoheadrightarrow x_2, y_1 \rightsquigarrow y_2)/D$. We get $\mathbf{a} \to' \mathbf{a} = (b,1)/D \to' (b,1)/D = (b \twoheadrightarrow b, 1 \to 1)/D = (1,1)/D = \mathbf{1} = \mathbf{a} \rightsquigarrow' \mathbf{a}$. Analogously, $\mathbf{a} \to' \mathbf{b} = (b,1)/D \to' (1,a)/D = (a,a)/D = \mathbf{e} = \mathbf{a} \rightsquigarrow' \mathbf{b}$. Finally, we conclude that $\to' = \rightsquigarrow'$ and obtain the following table:

$\to'$	**a**	**b**	**c**	**d**	**e**	**1**
a	**1**	**e**	**b**	**a**	**c**	**d**
b	**a**	**1**	**a**	**d**	**d**	**1**
c	**1**	**d**	**1**	**a**	**a**	**d**
d	**d**	**c**	**e**	**1**	**b**	**a**
e	**d**	**a**	**d**	**1**	**1**	**a**
1	**a**	**b**	**c**	**d**	**e**	**1**

.

Thus, by Corollary 4.29, $\mathcal{A}/D := (A/D, \to', \to', D)$ is a (pseudo-) BCI algebra.

Theorem 4.31
Let $\mathcal{A}$ and $\mathcal{B}$ be pseudo-RM algebras and $\varphi\colon A \to B$ be a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. If $\mathcal{A}$ satisfies (pD) *and $\mathcal{B}$ satisfies* (An), *then $\mathcal{A}/\ker\varphi$ is isomorphic to $\mathcal{B}$.*

Proof. By Proposition 4.14, $D := \ker\varphi$ is a closed t-deductive system of $\mathcal{A}$. Define a function $\psi\colon A/D \to B$ by $\psi(x/D) = \varphi(x)$ for all $x \in A$. We have

$$\begin{aligned} x/D = y/D &\Leftrightarrow x \to y, y \to x \in D \\ &\Leftrightarrow \varphi(x) \to \varphi(y) = 1 = \varphi(y) \to \varphi(x) \\ &\Leftrightarrow \varphi(x) = \varphi(y). \end{aligned}$$

This means that ψ is well defined and one-to-one. It is easy to see that ψ is a homomorphism from $\mathcal{A}/D$ onto $\mathcal{B}$. Thus ψ is an isomorphism from $\mathcal{A}/D$ onto $\mathcal{B}$.

Corollary 4.32
If $\mathcal{A}$ is a pseudo-BB algebra, then $\mathrm{K}(\mathcal{A})$ *is a closed t-deductive system of $\mathcal{A}$ and* $\mathcal{A}/\mathrm{K}(\mathcal{A})$ *is a pseudo-BCI algebra.*

Proof. Since $\mathcal{A}$ satisfies (pD) and (p^{**}), by Proposition 4.4, $\mathrm{K}(\mathcal{A})$ is a closed compatible deductive system of $\mathcal{A}$. From Theorem 4.10 and Corollary 4.29 it follows that $\mathrm{K}(\mathcal{A}) \in \mathrm{T}_{\mathrm{cl}}(\mathcal{A})$ and $\mathcal{A}/\mathrm{K}(\mathcal{A})$ is a pseudo-BCI algebra.

5. Conclusions

Pseudo-M algebras are a common generalization of pseudo-BCK, pseudo-BCI, pseudo-BCH, pseudo-BE and pseudo-CI algebras. In this paper, the notion of deductive system in a pseudo-M algebra is introduced and its elementary properties are investigated. Closed deductive systems are defined and studied. Deductive systems of direct products of pseudo-M algebras are described. The homomorphic properties of (closed) deductive systems are provided. The concepts of translation deductive systems and R-congruences in pseudo-M algebras are introduced and studied. It is shown that there is a bijection between translation deductive systems and R-congruences. Finally, the construction of quotient algebra $\mathcal{A}/D$ of a pseudo-M algebra $\mathcal{A}$ via a translation deductive system D of $\mathcal{A}$ is given.

As a direction of research, one could define various deductive systems in a pseudo-M algebra, and investigate the relationships between these. It will also be possible to study the pseudo-M algebras verifying the commutative property or verifying the implicative property. Another topic of research could be to introduce and investigate the notion of fuzzy pseudo-M algebra.

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References

[1] Borumand Saeid, Arsham, and Akbar Rezaei. "Quotient CI-algebras." *Bull. Transilv. Univ. Braşov Ser. III* 5(54), no. 2 (2012): 15-22. Cited on 94.

[2] Borzooei, Rajab Ali, et al. "On pseudo BE-algebras." *Discuss. Math. Gen. Algebra Appl.* 33, no. 1 (2013): 95-108. Cited on 94, 98 and 103.

[3] Chaudhry, Muhammad Anwar, and Hafiz Fukhar-ud-din. "Ideals and filters in BCH-algebras." *Math. Japon.* 44, no. 1 (1996): 101-111. Cited on 94.

[4] Dudek, Wiesław Aleksander, and Young Bae Jun. "Pseudo-BCI algebras." *East Asian Math. J.* 24, no. 2 (2008): 187-190. Cited on 93.

[5] Dymek, Grzegorz. "Atoms and ideals of pseudo-BCI-algebras." *Comment. Math.* 52, no. 1 (2012): 73-90. Cited on 101.

[6] Dymek, Grzegorz. "On compatible deductive systems of pseudo-BCI-algebras." *J. Mult.-Valued Logic Soft Comput.* 22, no. 1-2 (2014): 167-187. Cited on 104.

[7] Dymek, Grzegorz. "On pseudo-BCI-algebras." *Ann. Univ. Mariae Curie-Skłodowska Sect. A* 69, no. 1 (2015): 59-71. Cited on 98.

[8] Georgescu, George, and Afrodita Iorgulescu. "Pseudo-BCK algebras: an extension of BCK algebras." In: *Combinatorics, computability and logic. Proceedings of the Third International Conference on Combinatorics, Computability and Logic, (DMTCS'O1)*, 97-114. London: Springer-Verlag, 2001. Cited on 93.

[9] Hoo, Cheong Seng. "Filters and ideals in BCI-algebras." *Math. Japon.* 36, no. 5 (1991): 987-997. Cited on 94.

[10] Hu, Qing Ping, and Xin Li. "On BCH-algebras." *Math. Sem. Notes Kobe Univ.* 11, no. 2 (1983): 313-320. Cited on 93.

[11] Imai, Yasuyuki, and Kiyoshi Iséki. "On axiom systems of propositional calculi. XIV." *Proc. Japan Acad.* 42 (1966): 19-22. Cited on 93.

[12] Iorgulescu, Afrodita. *Algebras of logic as BCK algebras.* Bucharest: Editura ASE, 2008. Cited on 98.

[13] Iorgulescu, Afrodita. "New generalizations of BCI, BCK and Hilbert algebras – Part I." *J. Mult.-Valued Logic Soft Comput.* 27, no. 4 (2016): 353-406. Cited on 96.

[14] Iorgulescu, Afrodita. *Implicative-groups vs. groups and generalizations.* Bucharest: Matrix Rom, 2018. Cited on 94 and 95.

[15] Iséki, Kiyoshi. "An algebra related with a propositional calculus." *Proc. Japan Acad.* 42 (1966): 26-29. Cited on 93.

[16] Jun, Young Bae. "Some results on ideals of BCK-algebras." *Sci. Math. Jpn.* 53, no. 3 (2001): 477-480. Cited on 94.

[17] Kim, Hee Sik, and Young Hee Kim. "On BE-algebras." *Sci. Math. Jpn.* 66, no. 1 (2007): 113-116. Cited on 93.

[18] Kim, Young Hee, and Keum Sook So. "On minimality in pseudo-BCI-algebras." *Commun. Korean Math. Soc.* 27, no. 1 (2012): 7-13. Cited on 98.

[19] Kühr, Jan. *Pseudo-BCK-algebras and related structures.* Olomouc: Univerzita Palackého v Olomouci, 2007. Cited on 98 and 104.

[20] Meng, Biao Long. "CI-algebras." *Sci. Math. Jpn.* 71, no. 1 (2010): 11-17. Cited on 93 and 104.

[21] Meng, Biao Long. "On filters in BE-algebras." *Sci. Math. Jpn.* 71, no. 2 (2010): 201-207. Cited on 94.

[22] Meng, Biao Long. "Closed filters in CI-algebras." *Sci. Math. Jpn.* 71, no. 3 (2010): 367-372. Cited on 94 and 100.

[23] Meng, Jie. "On ideals in BCK-algebras." *Math. Japon.* 40, no. 1 (1994): 143-154. Cited on 94.

[24] Meng, Jie, and Shi Ming Wei. "Periodic BCI-algebras and closed ideals." *Math. Japon.* 38, no. 3 (1993): 571-575. Cited on 94.

[25] Rezaei, Akbar, Arsham Borumand Saeid, and Yousefi Sikari Saber. "On pseudo-CI algebras." *Soft Comput.* 23, no. 13 (2019): 4643-4654. Cited on 94, 98, 104 and 108.

[26] Roh, Eun Hwan, Seon Yu Kim, and Young Bae Jun. "On a problem in BCH-algebras." *Math. Japon.* 52, no. 2 (2000): 279-283. Cited on 106.

[27] Walendziak, Andrzej. "On normal filters and congruence relations in BE-algebras." *Comment. Math.* 52, no. 2 (2012): 199-205. Cited on 94.

[28] Walendziak, Andrzej. "Pseudo-BCH-algebras." *Discuss. Math. Gen. Algebra Appl.* 35, no. 1 (2015): 5-19. Cited on 94, 98, 100, 101 and 103.

[29] Walendziak, Andrzej. "On ideals of pseudo-BCH-algebras." *Ann. Univ. Mariae Curie-Skłodowska Sect. A* 70, no. 1 (2016): 81-91. Cited on 94 and 106.

[30] Walendziak, Andrzej. "Deductive systems and congruences in RM algebras." *J. Mult.-Valued Logic Soft Comput.* 30, no. 4-6 (2018): 521-539. Cited on 106.

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